e.g. $\mathcal{L} = \underline{\bar{\Psi} i \gamma^\mu \partial_\mu \Psi - m \bar{\Psi}\Psi}$

$\rightarrow$ global sym under $\Psi \rightarrow e^{i\alpha Q}\Psi$, $\alpha = $ const.

e.g. $\mathcal{L} = \bar{\Psi} i \gamma^\mu (\partial_\mu + ieQA_\mu)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

$\rightarrow$ "local symmetry" under $\begin{cases} \Psi \rightarrow e^{i\alpha Q}\Psi \\ A_\mu \rightarrow A_\mu - \frac{1}{e}\partial_\mu \alpha \end{cases}$, $\alpha = \alpha(x)$.

"U(1)" invariance

Generalize to bigger groups.

$SU(N) = N \times N$ unitary matrices with $\det = 1$.

$$\underbrace{U(\alpha^a)}_{N \times N \text{ matrices}} = e^{i\alpha^a t^a}, \quad t^a = \text{generators}, \quad \underbrace{[t^a, t^b] = i f^{abc} t^c}_{\text{don't commute}}.$$

$a = 1, 2, \ldots, N^2 - 1$

$\rightarrow$ "non-Abelian"

Let $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix}$ = column vector of fermions.

matrix notation

e.g. $\mathcal{L} = \underline{\bar{\Psi} i \gamma^\mu \partial_\mu \Psi - m \bar{\Psi} \Psi}$

$\rightarrow$ invariant under $\Psi \rightarrow U \Psi$, $U = U(\alpha^a)$, α^a = const.

$\rightarrow$ global $SU(N)$ symmetry

Now do local SU(N) (generalize QED)

"Non-Abelian gauge theory"

summed on a.

e.g. $\mathcal{L} = \bar{\Psi} i\gamma^\mu D_\mu \Psi - m\bar{\Psi}\Psi - \frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu}$

$D_\mu = \partial_\mu + i g \overbrace{A^a_\mu}^{N^2-1} t^a$

$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \underbrace{- g f^{abc} A^b_\mu A^c_\nu}$

$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix}$

gauge coupling

summed on $a = 1, 2, \ldots, N^2-1$

generalizes Q

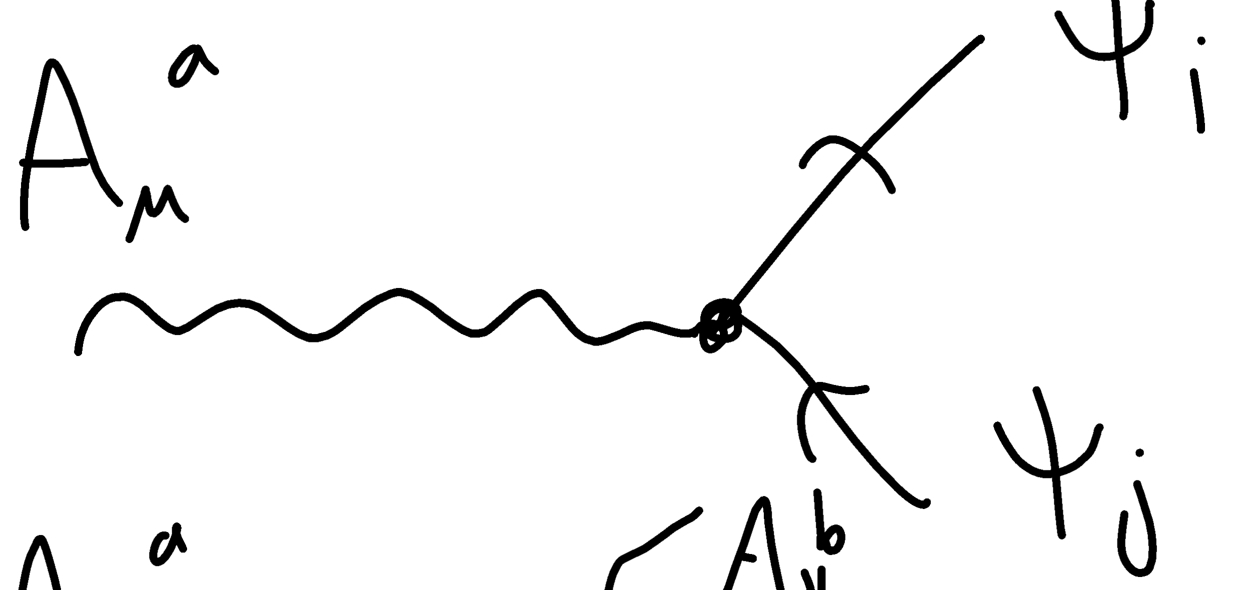
not present in QED

proportional to f^{abc}

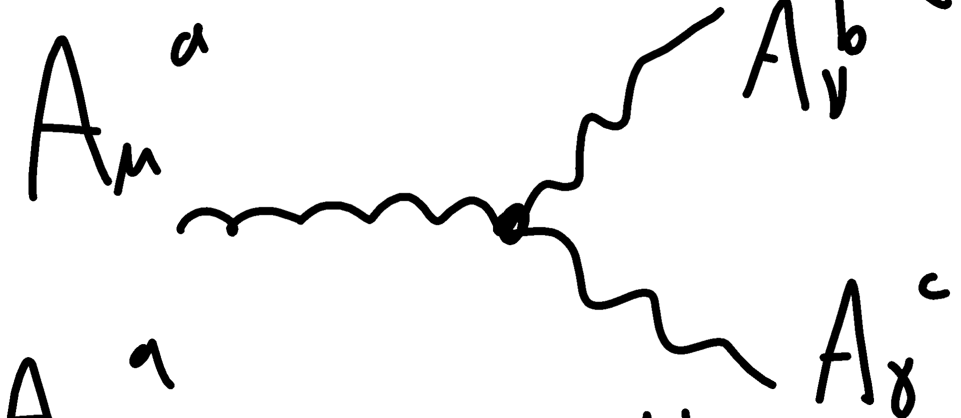
QFT recipe: N fermions = "N colours", "quarks"

N^2-1 vector bosons, massless, "gluons"

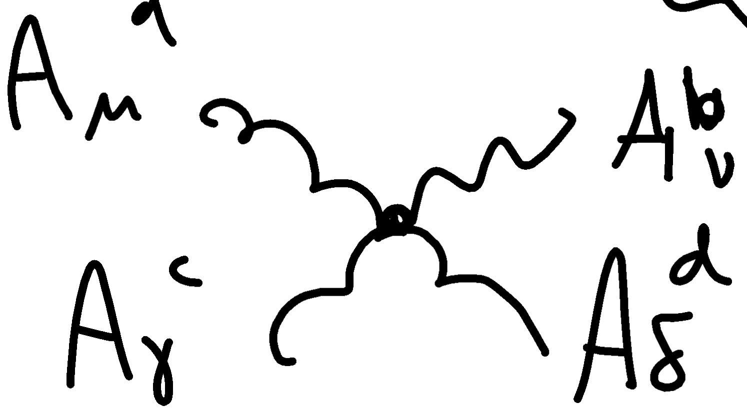
Interactions:



$= -i g t^a_{ij}$



$\propto g \cdot f^{abc} (\ldots)$



$\propto g^2 (f^{abc})^2$

QCD: $\mathcal{L}_{QCD} = \sum_q \left[\bar{q}\, i\gamma^\mu D_\mu q - m_q \bar{q} q \right] - \frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu}$

SU(3)

sum over quarks "flavours"

gluons

Spontaneous Symmetry Breaking

e.g. $\mathcal{L} = |\partial\phi|^2 - V(\phi)$, ϕ = complex scalar field

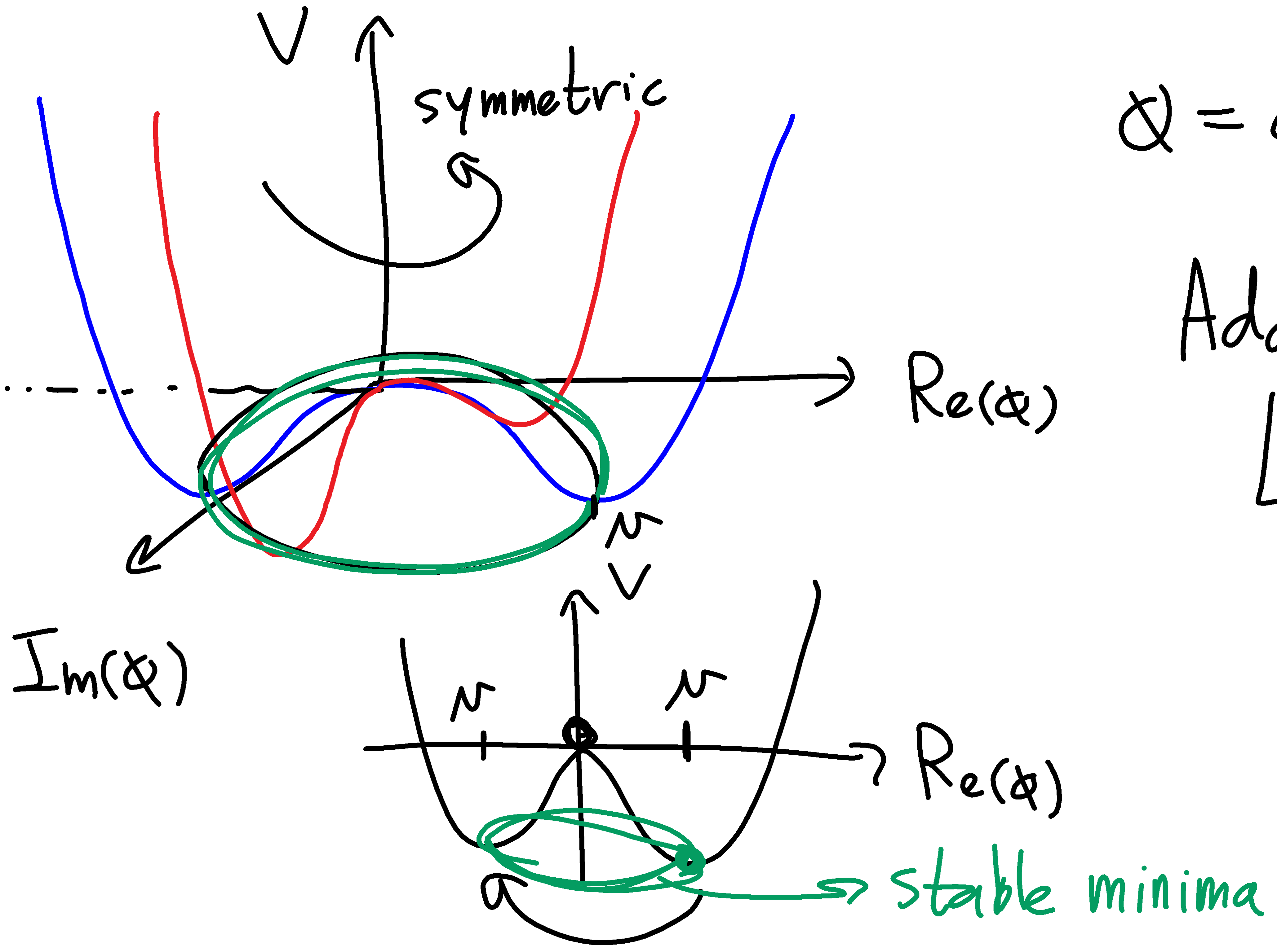
$V(\phi) = -\mu^2|\phi|^2 + \frac{\lambda}{2}|\phi|^4$

sign

$\hookrightarrow$ symmetry under $\phi \to e^{i\alpha}\phi$, α = const.

Try to interpret as mass term: $m_{phys} = \sqrt{-\mu^2} = i\mu$?

$$V(\phi) = -\mu^2 |\phi|^2 + \frac{\lambda}{2} |\phi|^4$$

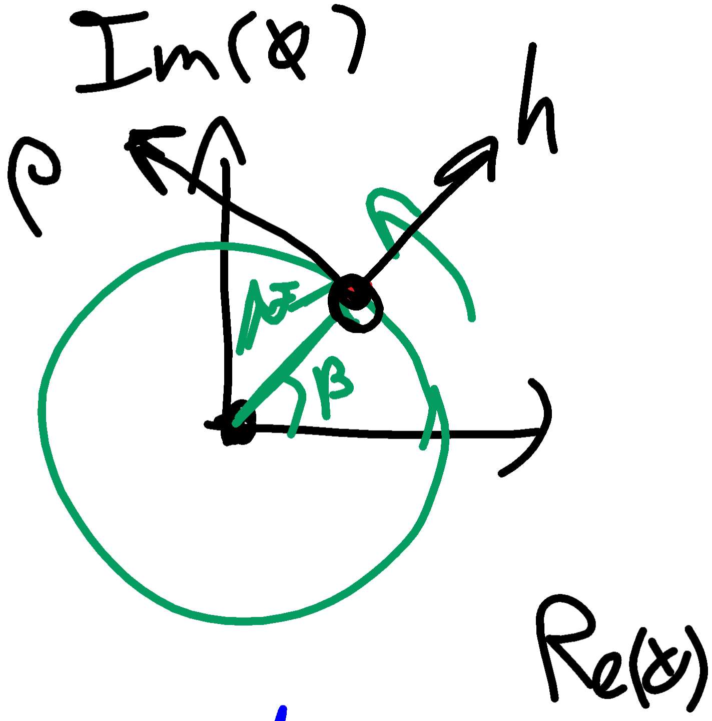


$\phi = 0$ is not a stable point!

Add QFT Rule 0.

↳ Expand around a stable point of the potential!

Minima: $\dfrac{\partial V}{\partial|\phi|} = 0 \Rightarrow |\phi| = \sqrt{\mu^2/\lambda} := v$



$\hookrightarrow \quad \phi = v e^{i\beta}, \quad \beta \in [0, 2\pi)$

$\hookrightarrow$ choose this to expand around

$$\phi(x) = \left[\underset{\text{min.}}{v} + h(x)/\sqrt{2}\right] e^{i\left[\beta + \rho(x)/\sqrt{2}v\right]}$$

min.

$h(x), \rho(x)$ are Real!

fluctuations around min.

Now plug back into $\mathcal{L}$.

$$\mathcal{L} = \frac{1}{2}(\partial h)^2 + \frac{1}{2}\left(1+\frac{h}{\sqrt{2}v}\right)^2(\partial \rho)^2$$

$$- \Big[\text{const.} + \frac{1}{2}\underbrace{(2\lambda v^2)}_{m_{phys}^2 = 2\lambda v^2}h^2 + \frac{\lambda}{\sqrt{2}}v h^3 + \frac{\lambda}{8}h^4\Big]$$

$\hookrightarrow$ mass of h.

but still there as $\rho \to \rho + \alpha$

$U(1)$ symmetry is not obvious.

h has mass $m_h = \sqrt{2\lambda}\cdot v$

ρ has mass $m_\rho = 0$.

$\hookrightarrow$ Nambu Goldstone boson

e.g. $\mathcal{L} = |\underbrace{(\partial_\mu + igQA_\mu)}_{\text{covariant deriv.}}\phi|^2 - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

$\rightarrow$ local $U(1)$ gauge invariance : $\begin{cases} \phi(x) \to e^{i\alpha(x)Q}\phi(x) \\ A_\mu \to A_\mu - \frac{1}{g}\partial_\mu\alpha \end{cases}$

$$V(\phi) = -\mu^2|\phi|^2 + \frac{\lambda}{2}|\phi|^4$$

Expand around $\underset{\langle\phi\rangle}{\overset{\shortparallel}{v e^{i\beta}}}$: $\phi(x) = \left(v + \frac{h}{\sqrt{2}}\right)e^{i(\beta+\rho/\sqrt{2}v)}$

$\rightarrow$ choose gauge such that $(\beta+\rho/\sqrt{2}v) = 0$, everywhere.

$\hookrightarrow \alpha(x) = \frac{-1}{Q}\left(\beta+\frac{\rho}{\sqrt{2}v}\right)$.

In this gauge, $\phi(x) = v + h(x)/\sqrt{2}$.

$$\mathcal{L} = |(\partial_\mu + igQA_\mu)(v + h/\sqrt{2})|^2 - V(h) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

↳ same as before

$$= \frac{1}{2}(\partial h)^2 + \frac{1}{2}\cdot 2g^2(v + h/\sqrt{2})^2 A_\mu A^\mu$$

kinetic for h

↳ mass term for A_μ !

"The Nambu Goldstone $\overset{\rho(x)}{\shortparallel}$" boson from SSB has been eaten".

h, real scalar with mass; m_h

A_μ = massive vector boson → 3 degrees of freedom
↳ $s=1$

But massless vector boson only has 2 degrees of freedom.

$$\underset{\phi}{2} + \underset{A_\mu \text{ massless}}{2} = \underset{h}{1} + \underset{A_\mu \text{ massive}}{3}$$ → "Higgs mechanism".

$$\Phi(x) = \left(v + h(x)/\sqrt{2}\right) \exp\left(i\left[\beta + \rho(x)/\sqrt{2}v\right]\right)$$

Now choose gauge transformation with $\alpha(x) = \frac{-1}{Q}\left(\beta + \frac{\rho}{\sqrt{2}v}\right)$

$$\Phi(x) \to e^{i\alpha(x)Q}\,\Phi(x) = v + h(x)/\sqrt{2}.$$

Chiral Fermions

$$\psi = \text{Dirac fermion} = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}\begin{matrix}\left.\vphantom{\begin{matrix}\psi_1\\ \psi_2\end{matrix}}\right\}\psi_L\\ \left.\vphantom{\begin{matrix}\psi_3\\ \psi_4\end{matrix}}\right\}\psi_R\end{matrix}$$

Can show that ψ_L, ψ_R transform independently under Lorentz.

$$\mathcal{L} = \bar{\psi} i\gamma^\mu \psi - m\bar{\psi}\psi = \overbrace{\bar{\psi}_L i\gamma^\mu \partial_\mu \psi_L + \bar{\psi}_R i\gamma^\mu \partial_\mu \psi_R}^{\text{independent } L,R}$$

$$- m\underbrace{(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L)}_{\text{mix } L,R}$$

The SM, finally

Gauge Group: $SU(3)_c \times \underbrace{SU(2)_L \times U(1)_Y}_{\text{"electroweak"}}$

"QCD" = "strong"

"electroweak" = "weak + electromagnetism"

Each generation:

$$Q_L = (3, 2, \tfrac{1}{6}) = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

$$u_R = (3, 1, +\tfrac{2}{3})$$

$$d_R = (3, 1, -\tfrac{1}{3})$$

quarks, transform under $SU(3)_c$.

$$L_L = (1, 2, -\tfrac{1}{2}) = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

$$e_R = (1, 1, -1)$$

leptons don't transform under $SU(3)_c$

$$H = \text{Higgs field} = \left(1, 2, \tfrac{1}{2}\right) = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix}$$

We also have vector bosons:

$$\left\{ \begin{array}{l} G_\mu^a = \text{gluons for } SU(3)_c \,;\; a = 1, 2, \ldots, \overset{N^2-1}{8} \\ W_\mu^p = \text{weak vectors for } SU(2)_L \,;\; p = 1, 2, 3 \\ B_\mu = \text{hypercharge } U(1)_Y . \end{array} \right.$$

$$\mathcal{L} = \mathcal{L}_{gauge}^{\text{"Yang-Mills"}} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa}$$

$$\mathcal{L}_{gauge} = -\frac{1}{4}\left(G_{\mu\nu}^{a}\right)^2 - \frac{1}{4}\left(W_{\mu\nu}^{p}\right)^2 - \frac{1}{4}\left(B_{\mu\nu}\right)^2$$

$$+ \bar{Q}_L i\gamma^\mu D_\mu Q_L + \bar{u}_R i\gamma^\mu D_\mu u_R + (d_R, L_L, e_R)$$

$$D_\mu Q_L = \partial_\mu + ig_s \underbrace{t_c^a}_{\text{"}3\times3\ SU(3)_c\text{ gen.}} G_\mu^a + ig \underbrace{t_L^p}_{\text{"}2\times2\ SU(2)_L\text{ gen.}} W_\mu^p + ig' \cdot \overset{\text{"}Y}{\left(\frac{1}{6}\right)} B_\mu$$

$$D_\mu e_R = \partial_\mu + \underset{\rightarrow}{0} \cdot \cancel{G_\mu^a} + \underset{\rightarrow}{0} \cdot \cancel{W_\mu^p} + ig'(-1) B_\mu$$

$$\mathcal{L}_{Yukawa} = \underbrace{-y_u \bar{Q}_L \underset{\substack{\| \\ i\sigma^2 H}}{\tilde{H}} u_R - y_d \bar{Q}_L H d_R - y_e \bar{L}_L H e_R + (h.c.)}_{\text{gauge invariant operators}}$$

$$\mathcal{L}_{Higgs} = |D_\mu H|^2 - \left(-\mu^2 |H|^2 + \frac{\lambda}{2} |H|^4 \right)$$

sign

$$D_\mu H = \partial_\mu + i g \underset{\substack{\| \\ \frac{\sigma^p}{2}}}{t_L^p} W_\mu^p + i g' \left(\tfrac{1}{2}\right) B_\mu$$

Higgs!

$$V(H) = -\mu^2 \overset{H^\dagger H}{\overset{\parallel}{|H|^2}} + \frac{\lambda}{2}|H|^4 \rightarrow |H|^2 = \mu^2/\lambda := v^2 = \text{min.}$$

Now choose a nice "unitarity" gauge.

$$\underset{\text{"Higgs field"}}{\underset{\parallel}{H(x)}} = \begin{pmatrix} 0 \\ v + h(x)/\sqrt{2} \end{pmatrix}$$

↳ "Higgs boson"

$$H \sim (1, \underbrace{2, \tfrac{1}{2}}_{\text{transforms under } SU(2)_L,\, U(1)_Y.})$$

$$SU(2) \times U(1)_Y \supset U(1)_{em}$$

leaves Higgs vacuum invariant

Generated by $\underbrace{t^3_L + Y}_{} = Q$

$$H: \quad \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2} \cdot \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$//$$

$$(t^3_L + Y)\begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ v \end{pmatrix}$$

$$V(H) \to (\text{const.}) + \frac{1}{2} m_h^2 h^2 + (\#) h^3 + (\#) h^4 .$$

$$|D_\mu H|^2 = \frac{1}{2} (\partial h)^2 + \frac{1}{2} \frac{v^2}{2} \Big(\underbrace{g^2 \left[(W_\mu^1)^2 + (W_\mu^2)^2 \right] + (-g W_\mu^3 + g' B_\mu)^2}_{\text{ew vector boson mass terms!}} \Big) + \dots$$

$$W_\mu^\pm = \frac{1}{\sqrt{2}} \left(W_\mu^1 \mp i W_\mu^2 \right)$$

$$\Rightarrow m_W^2 = \frac{g^2}{2} v^2 ,$$ massive $W^\pm$ gauge bosons.

$$\left(-gW_\mu^3 + g' B_\mu\right)^2 \cdot \frac{v^2}{2}$$

$$= \begin{pmatrix} W_\mu^3 & B_\mu \end{pmatrix} \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} \cdot \frac{v^2}{2}$$

$\hookrightarrow$ mass^2 matrix, $\det(M^2) = 0$.

$\therefore$

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} C_w & -S_w \\ S_w & C_w \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}, \qquad S_w = \sin\theta_w = \frac{g'}{\sqrt{g^2 + g'^2}}, \quad C_w = \cos\theta_w = \frac{g}{\sqrt{g^2 + g'^2}}$$

$\swarrow$ mass eigenstates

$m_A = 0$, A_μ = massless photon

$m_Z^2 = \frac{\bar{g}^2}{2} v^2$, $\bar{g}^2 = g^2 + g'^2$. = massive Z^0

Counting: $H \to 4$ $\xrightarrow{\text{Higgs}}$ $h \to 1$

$W_\mu^p \to 3 \cdot 2$ $\quad$ $W_\mu^\pm \to 2 \cdot 3$ s=1 massive vector

$B_\mu \to 1 \cdot 2$ $\quad$ $A_\mu \to 1 \cdot \cancel{3}\,2$

massless vector $\quad$ $Z_\mu \to 1 \cdot 3$

$$\underset{H}{4} + \underset{W^p}{6} + \underset{B}{2} = \underset{h}{1} + \underset{W}{6} + \underset{A}{\cancel{3}\,2} + \underset{Z}{3} \quad \checkmark$$

$$-\mathcal{L}_{\text{Yukawa}} = y_u \bar{Q}_L \tilde{H} u_R + y_d \bar{Q}_L H d_R + y_e \bar{L}_L H e_R$$

$$L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

$$H = \begin{pmatrix} 0 \\ v + h(x)/\sqrt{2} \end{pmatrix}$$

$$-\mathcal{L}_{\text{Yukawa}} = \overbrace{y_u v}^{m_u} \left(1+\frac{h}{\sqrt{2}v}\right) \bar{u}_L u_R + \overbrace{y_d v}^{m_d} \left(1+\frac{h}{\sqrt{2}v}\right) \bar{d}_L d_R$$

$$+ \underbrace{y_e v}_{m_e} \left(1+h/\sqrt{2}v\right) \bar{e}_L e_R$$

no ν_L mass!

h couples to fermions proportionally to their masses!

Dirac Mass: $m(\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L)$

mixes different Ψ_L, Ψ_R

With only Ψ_L, can write $-\mathcal{L}_M = M\,\Psi_L^t(-i\gamma^2)\Psi_L + (h.c.)$

$\hookrightarrow$ Majorana mass

Not invariant under $\Psi_L \to e^{i\alpha}\Psi_L$!

$(-i\gamma^2)\Psi_L^* =$ right-handed fermion

$$Q = t^3 + Y$$

↳ em transformation: $\psi \to e^{i\alpha Q}\psi$

$$\begin{cases} u_L : \; t^3_L = \frac{1}{2}, \; Y = \frac{1}{6} \Rightarrow Q = \frac{2}{3} \\ u_R : \; t^3_L = 0, \; Y = \frac{2}{3} \Rightarrow Q = \frac{2}{3} \\ d_R : \; t^3_L = -\frac{1}{2}, \; Y = \frac{1}{6} \Rightarrow Q = -\frac{1}{3} \end{cases}$$

$$e^{i\alpha Q} H = \left(1 + i\alpha Q + \cdots\right) H$$

$$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix}$$

$$Q = t^3 + Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$e^{i\alpha Q} = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix}$$

$$e^{i\alpha Q} \begin{pmatrix} 0 \\ v + h/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ v + h/\sqrt{2} \end{pmatrix}$$