TRISEP Notes on the Standard Model

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Thanks to lots of excellent work by many people over the past century, we have developed an amazingly successful picture of how regular stuff works at the subatomic level called the Standard Model, or "SM" for short. This theory gives a mathematical description of the fundamental constituents of Nature and the interactions between them. It has been tested in experiments to a greater precision than any other theory we know of, and aside from a few small but puzzling discrepancies, it passes with flying colours.

In these lectures, I will try to give you a big-picture overview of the Standard Model (SM). The time I have is much too short to go into full detail, but hopefully there will be enough information here to set you up to take a deeper study later on. A more detailed version of these notes can be found at Ref. [1]. I also recommend the excellent textbooks of Refs. [2, 3, 4, 5] and the notes of Refs. [6, 7].

These notes will be split into three main parts:

1. Quantum Field Theory, Symmetries, and Gauge Invariances
2. Spontaneous Symmetry Breaking and the Higgs Mechanism
3. Building Up the Standard Model

The first two parts will develop the tools that underlie the SM. I'll apply them to build up the SM itself. These tools should also be useful for the other theory-oriented lectures in this school. Some background and reference material is collected in a series of Appendices at the end.

Regarding notation, I will *natural units* such that

$$\hbar = c = 1 \ , \tag{1}$$

where $\hbar$ is the usual quantum mechanics thing and c is the speed of light. Since $\hbar$ has units of energy times time, $\hbar = 1$ implies that we are measuring time in units of inverse energy. Similarly, $c = 1$ means we are measuring distance in units of time and therefore in units of inverse energy as well. This simplifies dimensional analysis since now all dimensionful quantities can be expressed in units of energy. The specific unit I will use for energy is the electron Volt (eV), corresponding to the energy acquired by an electron passing through a potential difference of one Volt. Note also that $\text{keV} = 10^3\,\text{eV}$, $\text{MeV} = 10^6\,\text{eV}$, $\text{GeV} = 10^9\,\text{eV}$, and $\text{TeV} = 10^{12}\,\text{eV}$.

If you have any questions or find any typos, please feel free to email me.

1 QFT, Symmetries, and Gauge Invariances

Quantum Field Theory (QFT) is the structure out of which the SM is built. The best-tested theoretical predictions in science come from QFT calculations, and QFT is also main theoretical language used in many other fields of physics. It is a subject that should be covered as part of the standard undergraduate physics curriculum, but it rarely is. In this section I will give a very quick overview of the main elements of QFT that we need for the SM, with a specific focus on symmetries and gauge invariance.

1.1 QFT Intro

Quantum field theory is just regular quantum mechanics applied to continuous systems, in which the dynamical variables are *fields* that are functions of space and time. A specific example of a (classical) field theory that you've already worked with is electromagnetism, where the variables that you solve for are the electric and magnetic fields $\vec{E}(t,\vec{x})$ and $\vec{B}(t,\vec{x})$. When quantum mechanics is applied to fields, excitations of the fields can end up as the things we identify as particles. It might seem strange to describe discrete-looking things like particles in terms of continuous fields, but it works and there is a reason for it. In contrast to "regular" quantum mechanics where the number of particles is fixed, the QFT picture allows for any number of particles. This is needed to accommodate special relativity which allows particle creation and transmutation to be consistent with energy conservation.

1.1.1 Warmup: Special Relativity

Special relativity means that the laws of physics are unchanged under rotations or boosts. The fancy way of saying this is that nature is *invariant under Lorentz transformations*, which are general combinations of boosts and rotations. As long as gravity doesn't figure in, Lorentz invariance seems to be a symmetry of the universe.

To talk about special relativity, we need to specify how things change under Lorentz transformations. The most simple kinds of objects are *Lorentz scalars* that don't transform at all, such as the mass of a particle.[1] Next, we have Lorentz 4-vectors a^μ, $\mu = 0,1,2,3$ that transform like

$$a^\mu \to a^{\mu\,\prime} = \Lambda^\mu{}_\nu a^\nu = \Lambda^\mu{}_0\, a^0 + \Lambda^\mu{}_1\, a^1 + \Lambda^\mu{}_2\, a^2 + \Lambda^\mu{}_3\, a^3\ . \tag{2}$$

where $\Lambda^\mu{}_\nu$ is the 4×4 Lorentz transformation matrix. Note that here I'm using the Einstein summation convention where you sum implicitly over any repeated index.[2] Two very important kinds of 4-vectors are position,

$$x^\mu = (t,x,y,z)\ , \tag{3}$$

[1] If somebody ever told you that particles "get heavier" when they go faster, ignore it. They don't.
[2] Apparently he considered this his greatest contribution to physics.

and momentum

$$p^\mu = (E, p^x, p^y, p^z) \ , \tag{4}$$

where E is the energy. For a particle of mass m, the energy is equal to $E = \sqrt{m^2 + \vec{p}^2}$. When $\vec{p} = 0$, corresponding to the particle at rest, this reproduces Einstein's famous formula $E = m$ (remembering that we have natural units with $c = 1$.) Sometimes we call the spatial components of a 4-vector a 3-vector, such as $p^\mu = (E, \vec{p})$.

Given any pair of 4-vectors, we can make a Lorentz scalar by connecting them with the 4×4 matrix $\eta_{\mu\nu}$ according to

$$a \cdot b \ \equiv \ a^\mu b^\nu \eta_{\mu\nu} \ = \ a^0 b^0 - \vec{a} \cdot \vec{b} \ , \tag{5}$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ . \tag{6}$$

It is also convenient to define 4-vectors with lower indices by

$$a_\mu \ \equiv \ \eta_{\mu\nu} a^\nu \tag{7}$$

$$= \ \begin{cases} a^0 & ; \ \mu = 0 \\ -a^i & ; \ \mu = i = 1, 2, 3 \end{cases} \tag{8}$$

We can also raise indices with $\eta^{\mu\nu}$, which is a 4×4 matrix with the same elements as $\eta_{\mu\nu}$. A very important lower-index object is the derivative operator,

$$\partial_\mu \ = \ (\partial/\partial t, \, \partial/\partial x, \, \partial/\partial y, \, \partial/\partial z) \ = \ (\partial_t, \, \vec{\nabla}) \ . \tag{9}$$

More generally, a *Lorentz tensor* is any object with multiple indices such that it transforms with a $\Lambda^\mu{}_\nu$ matrix for each index. For example,

$$T^{\mu\nu} \to T^{\mu\nu\,\prime} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta} \ . \tag{10}$$

In general, any object with all its indices contracted is *Lorentz invariant*, meaning that it is a Lorentz scalar (even if its individual parts are not).

1.1.2 Warmup: the Simple Harmonic Oscillator

This system is defined by the Hamiltonian

$$\mathscr{H} = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 \ , \tag{11}$$

where m is the particle mass and $m\omega^2 = k$ is the "spring constant". Classically, the equations of motion give

$$\ddot{x} = -\omega^2 x \ , \tag{12}$$

with the solution

$$x(t) = A\,\cos(\omega\,t + B)\;, \quad p(t) = m\,\dot{x}(t)\;, \tag{13}$$

for some constants A and B that we fix with initial conditions. Note that we can rewrite this classical solution in the form

$$x(t) = a\,e^{-i\omega\,t} + a^{\dagger}\,e^{i\omega\,t}\;, \tag{14}$$

with $a(t) = (A/2)\,e^{-iB}$.

Using the same Hamiltonian to define a quantum system, where now x and p are operators, we can built the eigenstates and eigenfunctions of the Hamiltonian out of the lowering and raising operators a and $a^{\dagger}$ that satisfy

$$[a,\,a] = 0 = [a^{\dagger},\,a^{\dagger}]\;, \quad [a,\,a^{\dagger}] = \omega\;. \tag{15}$$

We also have (in the Heisenberg picture with time-dependent operators)

$$x = a\,e^{-i\omega\,t} + a^{\dagger}\,e^{i\omega\,t}\;, \quad p = \frac{m}{i}\left(a\,e^{-i\omega\,t} - a^{\dagger}\,e^{i\omega\,t}\right) \tag{16}$$

The energy eigenstates and eigenvalues are

$$|n\rangle = \left(a^{\dagger}\right)^{n}|0\rangle\;, \quad E_n = \omega\,(n+1/2)\;, \tag{17}$$

where the vacuum $|0\rangle$ is annihilated by the lowering operator, $a|0\rangle = 0$.

1.1.3 Defining a QFT with an Action

We turn next to Quantum Field Theory (QFT), which is just regular quantum mechanics applied to continous *field* systems. In the one-particle quantum mechanics you might be more familiar with, the starting point is usually a set of variables that describe the system (such as x and p) and a Hamiltonian built from them that defines their time evolution. For QFT, we start instead with a set of field variables $\phi_i(t,\vec{x})$ defined over all of spacetime together with an action built from them of the form

$$S = \int d^4x\;\mathscr{L}(\phi_i)\;, \tag{18}$$

where $d^4x = dt\,dx\,dy\,dz$. Recall from your classical mechanics course that given an action, you can identify the conjugate variables and build a Hamiltonian, so this starting point for QFTs is basically just a generalization of what you've seen. Note, however, that t and $\vec{x}$ are now just spacetime labels rather than the degrees of freedom.

With the SM in mind, we want to study quantum field theories with Lorentz invariance built in from the start. This will be guaranteed as long as the Lagrangian itself is a Lorentz scalar. To make sure this is the case, we will always use field variables that have well-defined Lorentz transformation properties such that we can combine them into Lorentz-invariant terms in the Lagrangian. The three main classes of fields we need for the SM are scalar, fermion, and vector. For a Lorentz transformation Λ that takes $x \to x' \equiv \Lambda\,x$, we have:

Scalar: $\phi'(x') = \phi(x)$
Fermion: $\psi'_a(x') = M_a{}^b(\Lambda)\,\psi_b(x)$ (The indices here are *spinor* indices.)
Vector: $A'_\mu(x') = \Lambda_\mu{}^\nu A_\nu(x)$

The scalar and vectors shouldn't be too surprising. For the fermion, the different spinor components of the field transform amongst themselves through the matrix $M_a{}^b(\Lambda)$ determined by the Lorentz transformation Λ. These transformations give what is called a *representation* of the Lorentz group, and satisfy $M(\mathbb{I}) = \mathbb{I}$ and $M(\Lambda)M(\Lambda') = M(\Lambda\,\Lambda')$ for any Lorentz transformations Λ and Λ'. Beyond Lorentz invariance, the action should satisfy a few other basic conditions to describe a realistic theory: it should be real (for a "unitary" theory), it should depend on fields all evaluated at the same spacetime point (for a "local" theory), and all the fields and their derivatives should go to zero at spacetime infinity.

Once we have a set of fields and a nice action, we can get to applying quantum mechanics to it. I don't have time to get into the details of this, so I will only summarize the main results through a QFT recipe and some examples below. It turns out that we only know how to solve things exactly in certain very special cases. For everything else, the most common approach is to use perturbation theory to expand around the limit of the theory where there are no interactions. This leads to a set of computational rules that can be expressed in terms of *Feynman diagrams*. There are three steps, which we formulate as a set of three steps.

Recipe:

1. Start with the quadratic (and lower) terms in the Lagrangian and extract from them the kinetic and mass terms.
2. To do this, redefine the field variables to put the kinetic terms in *canonical* form and diagonalize the mass matrices.
3. Add the terms higher than quadratic (in terms of the redefined and now-canonical/diagonal fields) and identify the interactions they correspond to.

We will illustrate these recipe steps below, starting with the first two for scalar, fermion, and vector fields. With this in place, we will add interactions and discuss how to compute things using Feynman rules.

1.1.4 Free Fields

The first two steps in our QFT recipe correspond to identifying the particles in the theory and their mass terms from the action in the limit that we neglect interactions between them. I will illustrate this with a few examples.

e.g. Non-interacting real scalar

The basic Lagrangian for a real scalar is [2]

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \ , \tag{19}$$

where $(\partial\phi)^2 = \eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$. The first piece here is the kinetic term and the second is the mass term. This Lagrangian describes a spin $s = 0$ particle with mass $m = \sqrt{m^2}$. As per the steps above, the absence of terms higher than quadratic implies there are no interactions between the particles.

This simple theory can be solved exactly. The result is that the field takes the form

$$\phi(x) = \int \widetilde{dk}\ \left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x}\right)\ , \tag{20}$$

where $k^0 = \sqrt{m^2 + \vec{k}^2}$, $\widetilde{dk} \equiv d^3k/2k^0(2\pi)^3$, and the operators a_k and $a_k^\dagger$ satisfy

$$[a_k, a_p] = 0 = [a_k^\dagger, a_p^\dagger]\ , \quad [a_k, a_p^\dagger]\ \propto\ \delta^{(3)}(\vec{k} - \vec{p})\ . \tag{21}$$

Up to a constant, the Hamiltonian of the theory is

$$\mathscr{H} = \int \widetilde{dk}\ k^0\, a_k^\dagger a_k\ . \tag{22}$$

The Hilbert space consists of a vacuum state $|0\rangle$ with $a_k|0\rangle = 0$ together with excited states built by applying one or more $a^\dagger$ operators to it, such as $a_k^\dagger|0\rangle$ with energy $E = k^0$ and $a_k^\dagger a_p^\dagger|0\rangle$ with energy $E = (k^0 + p^0)$. All of this should remind you of the simple harmonic oscillator discussed above. The key difference here is that there is an independent oscillator for each momentum 3-vector $\vec{k}$. The ground state is interpreted as having no particles, while the excited states have as many particles as there are raising operators acting on it: $a_k^\dagger|0\rangle$ is one particle with momentum $\vec{k}$, and $a_k^\dagger a_p^\dagger|0\rangle$ is two particles with momenta $\vec{k}$ and $\vec{p}$.

Exercise: *show that the expansion of Eq.* (20) *satisfies the classical equation of motion* $(\partial^2 + m^2)\phi = 0$.
Exercise: *show that the energy of the two-particle state is consistent with the fact that the two particles don't interact with each other.*

e.g. **Non-interacting complex scalar**

For a complex scalar field Φ, the basic Lagrangian is

$$\mathscr{L} = |\partial\Phi|^2 - m^2|\Phi|^2 \tag{23}$$

where $|\partial\Phi|^2 = \eta^{\mu\nu}\partial_\mu\Phi^\dagger\partial_\nu\Phi$. As before, the first term is the kinetic term, the second is the mass term, and there are no interactions because everything is quadratic. This theory now describes two spin $s = 0$ particles. In this form, we usually call one of them a particle and the other its antiparticle, but this doesn't mean much without interactions.

The solution to this simple theory has the field expansion

$$\Phi(x) = \int \widetilde{dk}\ \left(a_k e^{-ik\cdot x} + b_k^\dagger e^{ik\cdot x}\right)\ , \tag{24}$$

where now a_k and b_k are independent sets of lowering operators that both annihilate the vacuum state $|0\rangle$. They each satisfy the same commutators as for the real scalars while also commuting with each other (including $[a_k, b_p^\dagger] = 0$). The Hamiltonian for the theory in terms of them is (up to a constant)

$$\mathscr{H} = \int \widetilde{dk}\; k^0 \left(a_k^\dagger a_k + b_k^\dagger b_k\right) \tag{25}$$

The interpretation is that $a_k^\dagger$ creates particles with 4-momentum $(k^0, \vec{k})$ and $b_k^\dagger$ antiparticles with the same 4-momentum. Note that both have the same mass m.

Exercise: *rewrite the Lagrangian in terms of the two real scalars ϕ_1 and ϕ_2 defined by $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$.*
Exercise: *show that the complex form we started with implies that the ϕ_1 and ϕ_2 must have the same mass.*

e.g. Non-interacting massive vector

Going next to (massive) vector fields, we run into a minor complication. We would like to use these fields to describe particles with spin $s = 1$. However, a spin-1 object has three components while the vector field A^μ seems to have four. What is happening is that a general vector field describes a spin-1 particle together with a spin-0 particle. Since we already know how to describe spin-0 particles with real scalars, we will impose a constraint on the vector to get rid of the unwanted spin-0 part, namely

$$\partial_\mu A^\mu = 0\ . \tag{26}$$

Taking the constraint into account, the basic Lagrangian is

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu\ , \tag{27}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is called the field strength.

As before, the solution to the quantum theory involves expanding the vector field as

$$A^\mu(x) = \sum_{\lambda=1}^{3} \int \widetilde{dk}\; \left[\epsilon^\mu(k,\lambda)\, a_{k,\lambda}\, e^{-ik\cdot x} + \epsilon^{\mu\,*}(k,\lambda)\, a_{k,\lambda}^\dagger\, e^{ik\cdot x}\right]\ . \tag{28}$$

This looks a lot like the real scalar, but now with some extra structure. In particular, there is a sum over three polarization states λ that label the spin together with the polarization vector $\epsilon^\mu(k,\lambda)$. These can be taken to satisfy

$$k \cdot \epsilon(k,\lambda) = 0\ ,\quad \epsilon(k,\lambda)\cdot\epsilon^*(k,\lambda') = -\delta_{\lambda\lambda'}\ . \tag{29}$$

We also have independent raising and lowering operators for each momentum mode and each polarization state. The interpretation is that $a_{k,\lambda}^\dagger$ creates a spin-1 particle with mass m, energy $E = k^0 = \sqrt{m^2 + \vec{k}^2}$, and polarization (spin) state λ.

Exercise: *show that the expansion of Eq.* (28) *satisfies the constraint $\partial_\mu A^\mu = 0$.*

***e.g.* Non-interacting Dirac fermion**

We turn next to fermion fields to describe particles with spin $s = 1/2$. These transform under Lorentz in a well-defined way through matrices $M(\Lambda)$ that are functions of the Lorentz transformation matrix Λ on vectors. For a Dirac fermion, the basic field object ψ is a four-component column vector with complex entries. The basic Lagrangian is

$$\mathscr{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \ . \tag{30}$$

Here, the γ^μ refer to the 4×4 gamma matrices that satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, and $\bar{\psi} = \psi^\dagger\gamma^0$ is four-component row vector. See Appendix A for more details. Even though it's not obvious, this Lagrangian is invariant under Lorentz transformations. It describes a spin-1/2 fermion and a spin-1/2 antifermion, both with mass m.

Similar to the previous examples, the fermion field can be expanded as a sum over modes:

$$\psi(x) = \sum_{s=1}^{2} \int \widetilde{dk} \ \left[u(k,s)\, a_{k,s}\, e^{-ik\cdot x} + v(k,s)\, b^\dagger_{k,s}\, e^{ik\cdot x} \right] \ , \tag{31}$$

where s runs over spin states, $u(k,s)$ and $v(k,s)$ are 4-component column vectors, and $a^\dagger_{k,s}$ ($b^\dagger_{k,s}$) is a raising operator that creates a fermion (anti-fermion) of momentum $\vec{k}$ and spin state s. In contrast to before, however, these operators satisfy anticommutation relations instead of commutation relations,

$$\{a_{k,s}, a_{p,s'}\} = \{a^\dagger_{k,s}, a^\dagger_{p,s'}\} = 0 \ , \quad \{a_{k,s}, a^\dagger_{p,s'}\} \ \propto \ \delta_{s,s'}\, \delta^{(3)}(\vec{k} - \vec{p}) \ . \tag{32}$$

Having anticommutators in place of commutators is characteristic of fermions and implies that you can't put more than one identical fermion into the same state.

Exercise: *prove this last point by building a state with two identical fermions having the same spin and momentum and using the anti-commutation relations to show that it is zero.*

1.1.5 Adding Interactions

Having covered the basic free scalar, vector, and fermion theories, we turn next to treating interactions. These come from terms in the Lagrangian that are higher than quadratic order in the fields. In general, an interaction term in the Lagrangian that is a product of N fields will describe an interaction between a total of N (incoming or outgoing) particles. As before, I'll illustrate through examples.

***e.g.* Real scalar with $\lambda\phi^4$ interaction**

Let's expand our real scalar theory with a new term that goes beyond quadratic order in the fields,

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\,\phi^4 \ . \tag{33}$$

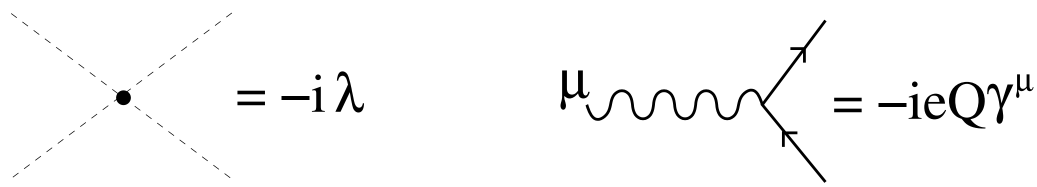

Figure 1: Feynman interaction vertices for the $\lambda\,\phi^4$ theory (left) and QED (right).

The new quartic term defines an interaction between four ϕ scalar particles. The quantity λ is called the *coupling constant* for the interaction, and the larger its value the stronger the interaction. As long as $\lambda \ll 1$ is not too big, we can treat the interaction as a perturbation on the free theory and expand in powers of the coupling. This corresponds to a form of time-dependent perturbation theory.

The expansion in powers of the coupling can written in a diagrammatic form using what are called Feynman rules. For scattering, these work as follows:

1. Each incoming or outgoing particle has a corresponding *external leg* with one end not attached to anything. Time goes from left to right (in my convention), so legs on the left refer to incoming states and legs on the right to outgoing states.

2. Except for the external legs, all other legs must connect to vertices. Diagrams can also have internal legs, corresponding to the exchange of intermediate virtual particles that don't appear in the initial or final states. Sometimes this produces closed loops in which the internal spins and momenta should be summed over.

3. All possible diagrams should be included to get the full answer. However, for small couplings it is usually a good approximation to keep only the first few diagrams with the least number of factors of λ.

The Feynman rules then prescribe a numerical value to each diagram. Adding them up gives the scattering amplitude (times $-i$). The Feynman rule for the value of each $\lambda\phi^4/4!$ vertex in a diagram is given in the left panel of Fig. 1. A more complete list of Feynman rules for this theory and an example are given in Appendix B.

e.g. **Interactions in Quantum Electrodynamics**

Quantum electrodynamics (QED) is the fancy name we give to the quantized version of electromagnetism. It is a quantum theory with a vector field $A^\mu = (\phi, \vec{A})$ that represents the photon and which can be identified with the scalar and vector potentials of electromagnetism in the classical limit. The theory also describes charged elementary particles. For now, we will assume that the electron is the only such particle. It has spin-1/2 and is described by a Dirac fermion field ψ. The interaction term in QED connecting the photon to the electron is

$$\mathscr{L} \supset -e\,Q\,\bar{\psi}\gamma^{\mu}\psi\,A_{\mu}\;, \tag{34}$$

with $Q = -1$ being the electron charge in units of e. The Feynman rule vertex corresponding to this interaction is shown in the right panel of Fig. 1. Here, the curly line represents the photon and the straight lines with arrows describe the electron or anti-electron. A full set of Feynman rules for QED is given in Appendix B, and I will discuss the QED Lagrangian a bit later.

More generally, there is a simple prescription to obtain the Feynman rule for a given vertex from a term in the Lagrangian as long as no derivatives are involved.[3] The vertex with n_ϕ ingoing or outgoing ϕ scalars, n_ψ ingoing ψ fermions (or outgoing antifermions), $n_{\bar{\psi}}$ outgoing ψ fermions (or ingoing antifermions), and n_A ingoing or outgoing A^μ vectors is

$$\text{Vertex} = i \times \frac{\partial^{(n_\phi+n_\psi+n_{\bar{\psi}}+n_A)}}{(\partial\phi)^{n_\phi}(\partial\psi)^{n_\psi}(\partial\bar{\psi})^{n_{\bar{\psi}}}(\partial A_\mu)^{n_A}}\,\mathscr{L}\,\Bigg|_{\phi=\psi=\bar{\psi}=A=0} \tag{35}$$

This looks messy, but it is simple to do in practice. It can be derived from the path integral formulation of QFT [**?**, **?**]. Note that ψ and $\bar{\psi}$ should be treated as independent variables.

Exercise: *derive the vertex values for $\lambda\phi^4$ and QED using the Lagrangian terms above and this formula. It looks messy, but it's really not too bad once you work through it.*

1.2 Global Symmetries

A symmetry is a transformation on the variables of a theory that does not change the underlying physics. For quantum field theories, this usually means that a transformation of the fields (and possibly of spacetime) leaves the form of the action unchanged. A symmetry is said to be *global* if the transformation changes the fields in the same way at all spacetime points. It is also useful to classify symmetries according to whether they are discrete or continuous. I'll explain what this means below.

1.2.1 Discrete Symmetries

A discrete symmetry is one in which the transformations can never be reduced smoothly to zero (by which I mean doing no transformation at all).

[3]If there are derivatives, there is a straightforward generalization.

e.g. $\lambda\phi^4$
This theory, given by the Lagrangian of Eq. (33), has a symmetry under $\phi(x) \to -\phi(x)$ in that this transformation doesn't change the action (or the Lagrangian). The physical implication of the symmetry is that an initial state with an even number of ϕ particles can only scatter into a final state with an even number ϕs, and similarly for odd to odd. For instance, $\phi + \phi \to \phi + \phi$ is possible but $\phi + \phi \to \phi + \phi + \phi$ is not. This might be obvious from the form of the vertex in the theory, but there is also a nice symmetry reason for it.

e.g. **Chiral Symmetry**
Consider the Lagrangian with real scalar ϕ and fermion ψ

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \bar{\psi}i\gamma^\mu\partial_\mu\psi - y\,\phi\,\bar{\psi}\psi\ . \tag{36}$$

This has a symmetry under

$$\phi \to -\phi\ , \quad \psi \to \gamma^5\psi\ , \quad \bar{\psi} \to -\bar{\psi}\gamma^5\ . \tag{37}$$

The implication of this symmetry is that a fermion mass term is not allowed in the Lagrangian since $\bar{\psi}\psi \to -\bar{\psi}\psi$ is not invariant under the transformation. The interaction term here connecting the scalar with the fermion is called a *Yukawa interaction*,[4] and the symmetry is called a chiral symmetry.

1.2.2 Continuous Symmetries

Like the name suggests, continuous symmetries are symmetries that can be described by one or more continuous parameters. A familiar example from classical mechanics are rotations, which can be specified by a set of rotation angles.

e.g. **Two real scalars**
Consider a theory with two real fields ϕ_1 and ϕ_2 and the Lagrangian

$$\begin{aligned}\mathscr{L} &= \frac{1}{2}\left[(\partial\phi_1)^2 + (\partial\phi_2)^2\right] - \frac{1}{2}m^2\left(\phi_1^2 + \phi_2^2\right)\\ &= \frac{1}{2}(\partial\phi)^t(\partial\phi) - \frac{1}{2}m^2\phi^t\phi\ ,\end{aligned}$$

where $\phi = (\phi_1, \phi_2)^t$. This theory is symmetric under transformations of the form

$$\phi \to \phi' = \mathcal{O}\phi\ , \tag{38}$$

where $\mathcal{O}$ is any constant 2×2 orthogonal matrix satisfying $\mathcal{O}^t\mathcal{O} = \mathbb{I}$. Up to a few signs, any such matrix can be parametrized in terms of the single parameter α:

$$\mathcal{O} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\ . \tag{39}$$

[4]After H. Yukawa who first proposed the form to describe pions interactions with nucleons.

This one parameter $\alpha \in [0, 2\pi)$ therefore describes a symmetry of the theory (and an infinite number of symmetry transformations, one for each value of α). Note that as $\alpha \to 0$, the transformation matrix approaches the identity matrix, $\mathcal{O} \to \mathbb{I}$.

***e.g.* A Dirac fermion**
Next consider the theory

$$\mathscr{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi - M\,\bar{\psi}\psi \ . \tag{40}$$

It is not hard to check that a symmetry is

$$\psi \to e^{i\alpha}\psi \ , \quad \bar{\psi} \to e^{-i\alpha}\bar{\psi} \ , \tag{41}$$

for any constant α.

Exercise: *define $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$ and rewrite the 2-scalar Lagrangian in terms of Φ and Φ^*. With this, show that the field rotation symmetry of the real scalars is equivalent to a symmetry under $\Phi \to e^{i\alpha}\Phi$ and $\Phi^* \to e^{-i\alpha}\Phi^*$.*

Continuous symmetries have the mathematical structure of a *Lie group*, which are basically just groups whose elements can be specified (in local patches) by a set of real coordinates. I'll talk more about them later on. An enormously important result for theories with continuous symmetries is *Noether's theorem*.[5] It states that for every continuous symmetry of a system, there is a corresponding conservation law. For example, thanks to this theorem we now understand energy conservation as being the result of symmetry under time translations and momentum conservation as coming from invariance under spatial translations.

Exercise: *given a Lagrangian with a continuous symmetry, Noether's theory tells you how to build a* ***conserved current*** *$j^\mu = (\rho, \vec{j})$ satisfying the continuity equation $\partial_\mu j^\mu = 0$. Show that the continuity equation implies that the quantity*

$$Q \equiv \int d^3x\ \rho \tag{42}$$

is constant in time. To do so, use the continuity equation and the divergence theorem to show that $\partial_t Q = 0$ (with the implicit assumption that the fields and that make up j^μ vanish at the boundary). We call Q the conserved charge corresponding to the symmetry.

1.3 QED and Gauge Invariance

After going over some general features of QFTs, we'll turn now to a very important specific QFT, namely quantum electrodynamics (QED). The Lagrangian for the theory is

$$\mathscr{L} = \sum_i \left[\, \overline{\psi}_i i\gamma^\mu (\partial_\mu + ieQ_i A_\mu)\psi_i - m\overline{\psi}_i\psi_i \,\right] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \tag{43}$$

[5] First discovered by the great mathematician Emmy Noether.

where again $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The vector field here represents the photon while the Dirac fermions ψ_i correspond to charged particles with masses m_i and charges Q_i. For now, we'll assume there are only two such fermions, both with charge $Q = -1$: the electron with mass $m_e = 0.511$ MeV, and the muon with mass $m_\mu = 105.7$ MeV.

The theory has a continuous symmetry under the global transformations

$$\left\{ \begin{array}{ccc} \psi_i & \to & e^{iQ_i\alpha}\psi_i \\ A_\mu & \to & A_\mu \end{array} \right. \tag{44}$$

This works provided the transformation parameter α has the same value everywhere.

Consider next what happens if we allow the transformation parameter to vary over spacetime: $\alpha = \alpha(x)$. Doing so, we find that the transformation above is no longer a symmetry of the theory. In particular,

$$\overline{\psi_i} i\gamma^\mu \partial_\mu \psi_i \to \overline{\psi}_i i\gamma^\mu \partial_\mu \psi_i + \overline{\psi} i\gamma^\mu (iQ_i \partial_\mu \alpha)\psi_i \ . \tag{45}$$

Evidently the transformation of Eq. (44) is *not* a symmetry of the theory for non-constant parameters $\alpha(x)$ due to the derivative acting on it.

However, the theory is invariant under spacetime-dependent transformations of this form if the vector field also transforms according to:

$$\left\{ \begin{array}{ccc} \psi_i & \to & e^{iQ_i\alpha}\psi_i \\ A_\mu & \to & A_\mu - \frac{1}{e}\partial_\mu\alpha. \end{array} \right. \tag{46}$$

To see how this works, note that the combined transformations imply that

$$(\partial_\mu + ieQ_i A_\mu)\psi_i := D_\mu \psi_i \to e^{iQ_i\alpha} D_\mu \psi_i, \tag{47}$$

and therefore $\overline{\psi}_i i\gamma^\mu D_\mu \psi_i$ is invariant under the transformation for arbitrary $\alpha(x)$. The differential operator D_μ is sometimes called a *covariant derivative*. It is also not hard to check that this shift in the photon field does not alter the photon kinetic field:

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) \to (\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{1}{e}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\alpha = F_{\mu\nu} + 0 \ . \tag{48}$$

Thus, QED is invariant under the combined transformations of Eq. (46) for any reasonable arbitrary function $\alpha(x)$.

At first glance this invariance might just seem like a clever trick, but the river beneath these still waters runs deep. Thinking back to regular electromagnetism (of which QED is just the quantized version), one often deals with scalar and vector potentials. These potentials are not unique and are therefore not observable (for the most part). Instead, the true "physical" quantities are the electric and magnetic fields. The vector field A^μ in QED, corresponding to the photon, is identified with these potentials according to

$$A^\mu = (\phi, \vec{A}), \tag{49}$$

where ϕ and $\vec{A}$ are the usual scalar and vector potentials. This is justified by the equations of motion derived from the QED Lagrangian provided we also identify

$$F^{0i} = -E^i, \qquad F^{ij} = -\epsilon^{ijk} B^k, \tag{50}$$

with the electric and magnetic fields. With this identification, the transformations of Eq. (46) coincide with the usual "gauge" transformations you should have encountered in electromagnetism. Sometimes we call A^μ the gauge boson and the operation of Eq. (46) a gauge transformation.

Keeping in mind the story from electromagnetism, the interpretation of the quantum fields in QED is that ***only those quantities that are invariant under the transformations of Eq.*** (46) ***are physically observable.*** In particular, the vector field A_μ that represents the photon is not itself an observable quantity, but the gauge-invariant field strength $F_{\mu\nu}$ is. Put another way, the field variables we are using are redundant, and the transformations of Eq. (46) represent an *equivalence relation*: any two set of fields (ψ, A_μ) related by such a transformation represent the same physical configuration. Sometimes the invariance under Eq. (46) is called a *local* or *gauge symmetry*, but it is not really a symmetry at all. A true symmetry implies that different physical configurations have the same properties. Gauge invariance is instead a statement about which configurations are physically observable.

Gauge invariance is also sensible if we consider the independent polarization states of the photon, of which there are two. The vector field A_μ represents the photon, but it clearly has four independent components. Of these, one component (corresponding to configurations of the form $A_\mu = \partial_\mu \phi$ for some scalar ϕ) is already non-dynamical on account of the form of the vector kinetic term. Invariance under gauge transformations effectively removes the additional longitudinal polarization leaving behind only the two physical transverse polarization states. Note as well that if the photon had a mass term, $\mathscr{L} \supset m^2 A_\mu A^\mu/2$, the theory would no longer be gauge invariant. Instead, the longitudinal polarization mode would enter as physical degree of freedom. Turning the argument around, gauge invariance forces the photon to be exactly massless. This is the more modern view – we start with gauge invarince as the fundamental assumption and use it to build the rest of the theory. Here, we see that it fixes the photon-fermion interactions, illustrating why it is so powerful. We will see shortly that gauge invariance is even more powerful when the underlying symmetry transformations are more complicated.

Exercise: *work through the details above and show that the QED Lagrangian is indeed invariant under general transformations of the form of Eq.* (46)*.*

1.4 Non-Abelian Continuous Groups

We saw above that a key feature of QED is the underlying gauge invariance under which the fields corresponding to charged particles transformed according to

$$\psi \to e^{i\alpha Q}\, \psi \ . \tag{51}$$

To build up the full structure of the Standard Model (SM), we need to generalize this type of transformation to more complicated forms. Once we do, we'll see that QED emerges as a subset of the bigger family. Before getting to this physics, however, it will be helpful to make a brief mathematical detour.

In QED, the collection of all such transformations of the form of Eq. (51), with α taking on any real value whatsoever, has the mathematical structure of a *group*. A group is a collection of objects together with a multiplication rule such that:

1. multiplying two objects in the group produces another object in the group
2. the collection includes the identity element
3. for every object in the group there is an inverse element such that multiplying them together gives the identity

All three conditions are met for the set of all transformations of the form of Eq. (51). If we define $U(\alpha) = e^{i\alpha Q}$, we have:

$$U(\alpha_1)\, U(\alpha_2) = U(\alpha_1 + \alpha_2)\ , \quad U(0) = 1 = \text{identity}\ , \quad U(-\alpha) = U^{-1}(\alpha)\ . \tag{52}$$

The formal mathematical name given to this group of rephasings is $U(1)$, or the group of 1×1 unitary matrices. Note as well that the group has infinitely many elements, but they can all be parametrized by the single variable $\alpha \in \mathbb{R}$.

Exercise: *Show the points above explicitly. For $U^{-1}(\alpha)$, the inverse means that $U^{-1}(\alpha)\, U(\alpha) = U(\alpha)\, U^{-1}(\alpha) = 1$.*

An example of a more complicated continuous group that you have already seen is $SU(2)$, defined by the set of all 2×2 unitary matrices with determinant equal to unity.[6] Any such $SU(2)$ transformation can be written in the form

$$U(\{\alpha^a\}) \;=\; e^{i\alpha^a t^a} \;\equiv\; \sum_{n=0}^{\infty} \frac{(i\alpha^a t^a)^n}{n!}\ , \tag{53}$$

where $a = 1, 2, 3$ is summed over in the exponent, $\alpha^a \in \mathbb{R}$, and $t^a = \sigma^a/2$ are basically the Pauli sigma matrices: $\sigma^1 = \sigma_x$, $\sigma^2 = \sigma_y$, and $\sigma^3 = \sigma_z$. We call the t^a matrices the *generators* of the group. This kind of transformation should be familiar from spin in quantum mechanics. It is easy to see that among these transformations, $\mathbb{I} = U(0)$ is the identity, and that $U^{-1}(\alpha^a) = U(-\alpha^a)$ is the inverse of $U(\alpha^a)$. However, the fact that the product of any two of these transformations can still be written in the form of Eq. (53) is non-trivial and relies on the commutation relations of the $SU(2)$ generators,

$$[t^a, t^b] = i\epsilon^{abc} t^c\ , \qquad (SU(2)) \tag{54}$$

where ϵ^{abc} is completely antisymmetric with $\epsilon^{123} = 1$. Using this relation together with the Baker-Cambell-Hausdorff formula, it can be shown that for any α^a and β^a parameters there exists some γ^a parameters such that $U(\alpha^a)\, U(\beta^a) = U(\gamma^a)$, in the form of Eq. (53).

[6] Recall that unitary means that $U^\dagger = U^{-1}$.

Exercise: *Work out the commutation relations of Eq.* (54) *using what you know about Pauli matrices. Next, use the fact that the* α^a *are real and the* t^a *are Hermitian to show that* $U^\dagger(\alpha^a) = U^{-1}(\alpha^a) = U(-\alpha^a)$ *is unitary.*

Compared to $U(1)$, the group $SU(2)$ has two important differences. First, it now takes three real parameters to specify a group element of $SU(2)$ instead of the single one needed for $U(1)$. And second, the multiplication of two elements in $SU(2)$ does not commute in general, in contrast to $U(1)$ where it does. The fancy math words connected with this is that commuting groups are said to be *Abelian* while those in which some elements do not are called *non-Abelian*. Evidently, $U(1)$ is Abelian and $SU(2)$ is not.

To build up the SM, we need to go even further and extend $SU(2)$ to $SU(N)$, the group of $N \times N$ unitary matrices with unit determinant. Just like $SU(2)$, a general group element of $SU(N)$ can be written in the form

$$U(\alpha^a) \;=\; e^{i\alpha^a t^a} \;, \tag{55}$$

where now $a = 1, 2, \ldots\, (N^2 - 1)$, and the generators are more complicated $N \times N$ Hermitian matrices. In analogy with $SU(2)$, these satisfy a set of commutation relations of the form

$$[t^a, t^b] = i f^{abc} t^c \;, \tag{56}$$

where the constants f^{abc} can be taken to be real and completely antisymmetric. They are called the *structure constants* of the theory and define the commutation relations of the generators, called the *Lie algebra*.

In physics, groups usally emerge in the context of the symmetries of a system. With a bit of thought, you should be able to convince yourself that the symmetries of theory have the structure of a group. In the context of a QFT, this means that the collection of transformations on the fields that leaves the form of the Lagrangian invariant is a group. We saw this precisely in QED with (global) rephasings of charged fields corresponding to the group $U(1)$. The natural extension of this for the more complicated $SU(N)$ groups is that they act on sets of N fields according to

$$\psi(x) \;\to\; e^{i\alpha^a t^a}\,\psi(x) \;, \tag{57}$$

where $\psi = (\psi_1, \psi_2, \ldots, \psi_N)^t$ is an N-component column vector of fields.

Exercise: *Think for a bit and convince yourself that the symmetries of a theory have the structure of a group.*

Transformations of the form of Eq. (57) are called *representations* of the group. In this case, the group elements of $SU(N)$ are realized as linear operators (*i.e.* matrices) acting on the space of fields. Having such a linear representation makes it much easier to build a Lagrangian out of the fields that is invariant under the group transformation. More generally, a representation of a group G is a set of invertible matrices $\{M\}$, one for each group element $g \in G$, such that a two basic conditions are met:

1. $M(g)\,M(h) = M(g{\cdot}h)$ for any $g,\, h \in G$

2. $M(1) = \mathbb{I}$, the identity matrix

The size of the matrices is called the *dimension of the representation*.

For $SU(N)$, we defined the group itself in terms of an N-dimensional representation. However, it isn't the only representation of the group. The *trivial* representation of any group is to map every group element to unity (and every generator to zero). To find a representation of $SU(N)$, it is enough to find a set of matrices t^a_r that satisfy the commutation relations of the Lie algebra, Eq. (56). In fact, this is something you've probably already done for $SU(2)$ in quantum mechanics in the context of spin. Here, the generators correspond to the spin operators $S_x = t^1$, $S_y = t^2$, and $S_z = t^3$. The trivial representation corresponds to spin $s = 0$ and has $t^a = 0$ (which satisfies the commutators trivially!), and the defining representation with $t^a = \sigma^a/2$ gives spin $s = 1/2$. However, there are also higher spin-s represenations with dimension $d = 2s+1$. The same thing is true for general $SU(N)$ groups: there is the trivial representation and the defining N-dimensional representation, but there are also many others of higher dimension.

Exercise: *Prove that the trivial representation satisfies the two rules above for it to be a representation. Also, show that given a representation of a Lie algebra with real f^{abc} in terms of Hermitian matrices t^a_r, the matrices $-(t^a)^*$ also give a representation (called the conjugate representation).*

1.5 Non-Abelian Gauge Theories

With all that heavy lifting out of the way, we are now equipped to generalize QED, with its gauge invariance under $U(1)$, to more complicated non-Abelian continuous groups like $SU(N)$. These generalizations are called *non-Abelian gauge theories* and they form the basis of the Standard Model.

It is not too hard to write down a theory with a global symmetry under a non-Abelian group. For example, consider a set of N free Dirac fermions $\psi_1, \psi_2, \ldots, \psi_N$ with equal masses. Their Lagrangian is

$$\mathscr{L} \;=\; \sum_{i=1}^{N} \bar{\psi}_i (i\gamma^\mu \partial_\mu - m)\psi_i \;=\; \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \;, \tag{58}$$

where in the second equality we have gone to matrix notation with $\psi = (\psi_1, \ldots, \psi_N)^t$, $\bar{\psi} = (\bar{\psi}_1, \ldots, \bar{\psi}_N)$. In this form, it is easy to see that the theory is invariant under $\psi \to U\,\psi$ for any $N \times N$ unitary matrix U (that does not depend on spacetime). This implies that the theory is invariant under global $SU(N)$ transformations as well as $U(1)$ transformations of the form $U = e^{i\alpha}\,\mathbb{I}$.

To build a gauge theory based on the group $SU(N)$, we want to generalize the invariance of the Lagrangian of Eq. (58) under global $SU(N)$ to an invariance under spacetime-dependent $SU(N)$ transformations of the form $U(\alpha^a) = \exp(i\alpha^a t^a_r)$ with $\alpha^a = \alpha^a(x)$. The obstacle to achieving this with the Lagrangian of Eq. (58) is the derivative term, which can now act on the $U(\alpha^a)$ matrix that arises in the transformation as well as the field ψ, just like we saw in the QED discussion in Eq. (45). To fix this, the trick is to generalize the story from QED and add a matrix-valued vector field to build a covariant derivative.

Consider now a theory with an N-tuple of fermions ψ and an $N \times N$ matrix of vector fields $A_\mu = t^a A^a_\mu(x)$ (with $a = 1, \ldots, N^2-1$ summed over). We take their transformations under local $SU(N)$ with $U = U(\alpha^a) = \exp(i\alpha^a t^a)$ to be

$$\psi \;\to\; U\,\psi \tag{59}$$

$$A_\mu \;\to\; UA_\mu U^\dagger + \frac{1}{ig}U(\partial_\mu U^\dagger) \tag{60}$$

With these transformations and a bit of work, one can show that

$$\begin{aligned} D_\mu\psi &\equiv (\partial_\mu + igA_\mu)\psi \\ &\to \left[\mathbb{I}\,\partial_\mu + ig\left(UA_\mu U^\dagger + \frac{1}{ig}U\partial_\mu U^\dagger\right)\right]U\psi \\ &= U\left(\partial_\mu + igA_\mu\right)\psi\ . \end{aligned} \tag{61}$$

This *covariant derivative* generalizes what we had in QED, and the vector fields that make up part of it are called *gauge bosons*. As an operator, the covariant derivative transforms as $D_\mu \to UD_\mu U^\dagger$. Using this feature, we can write the following invariant Lagrangian terms for the fermions,

$$\mathscr{L} \;\supset\; \bar{\psi}i\gamma^\mu D_\mu\psi - m\,\bar{\psi}\psi\ . \tag{62}$$

This looks just like QED, but now everything is a row or column vector or a matrix.

So far, so good, but the fermion expression above also contains the vector field (within the covariant derivative), and we need a kinetic term for it if the theory is to make sense.[7] It's not obvious how to do this while maintaining local $SU(N)$ invariance, so I will basically just tell you the underlying trick and the final result. The trick is to notice that as an operator

$$[D_\mu, D_\nu] \;=\; ig\,t^a_r(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\,f^{abc}A^b_\mu A^c_\nu) \;\equiv\; ig\,t^a_r F^a_{\mu\nu}\ , \tag{63}$$

where the second equality defines $F^a_{\mu\nu}$, which generalizes the field strength we encountered in QED. This commutator operator also has a nice transformation under $SU(N)$, namely $[D_\mu, D_\nu] \to U[D_\mu, D_\nu]U^\dagger$. We can use this feature to make an invariant kinetic term:

$$\begin{aligned} \mathscr{L} &\supset -\frac{1}{2(ig)^2}tr([D_\mu, D_\nu][D^\mu, D^\nu]) \\ &= -\frac{1}{2(ig)^2}\,(ig)^2 F^a_{\mu\nu}F^{b\,\mu\nu}\,tr(t^a_r t^b_r) \\ &= -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu}. \end{aligned} \tag{64}$$

[7]If $A^a_\mu(x)$ is not a dynamical field, then what sets its value?

This has the same form as the QED vector boson kinetic term at quadratic order in the fields. The full Lagrangian of our non-Abelian gauge theory is just the sum of the terms in Eqs. (62,64).

Exercise: *Work out the details for the results above and show that: i) the covariant derivative acting on ψ transforms in the way given in Eq. (61); ii) the field strength derived from the commutator has the form of Eq. (63).*
Two hints: $\partial_\mu(UU^\dagger) = 0 = \partial_\mu(U^\dagger U)$, and work out the commutator $[D_\mu, D_\nu]$ by treating it as an operator acting on ψ.

Our nice, non-Abelian gauge theory can be generalized in lots of ways. We considered a single set of fermions ψ that transformed under the defining N-dimensional representation of $SU(N)$. However, as mentioned above there exist other representations of the group. For a set of fields χ that transform under a d-dimensional representation of $SU(N)$ with generators t^a_d, the covariant derivative for them takes the same form as before but now with the $N \times N$ matrices t^a replaced by the $d \times d$ matrices t^a_d,

$$D_\mu \chi = (\partial_\mu + igA^a_\mu t^a_d)\,\chi\ . \tag{65}$$

In general, a non-Abelian gauge theory can contain lots of different fields provided they each transform under a well-defined representation of the underlying gauge group and enter the Lagrangian as parts of gauge-invariant operators.

If you look carefully at the results above, you'll see that all the non-Abelian formulas reduce to Abelian ones for $f^{abc} \to 0$. Also, as $g \to 0$ the theory reduces to the form of the global $SU(N)$ theory of Eq.(58) plus a set of decoupled massless vectors. The combination of non-zero g and f^{abc} is therefore where the magic of non-Abelian gauge theories lies. Looking at Eq. (62), we see that gauge invariance fixes how the gauge-charged ψ fields interact with the vector boson. Turning next to Eq. (64), expanding out the field strengths in terms of the A^a_μ shows a really important new property of non-Abelian gauge theories: the vector bosons now interact with themselves, even in the absence of charged fields! The interaction vertices can be written in terms of Feynman rules, which we show in Fig. 12 in Appendix B. All these vertices are proportional to at least one power of both g and f^{abc}, and thus the fundamental self-interactions of the vector bosons are a direct result of the underlying non-commuting nature of the gauge group.

e.g. Quantum Chromodynamics = QCD

Quantum chromodynamics (QCD) is the strong force part of the SM. It is a non-Abelian gauge theory with gauge group $SU(3)$. The gauge fields (of which there are $8 = 3^2 - 1$ components) are called gluons G^a_μ. In addition, there are six fermionic quark fields, $q = u,\ d,\ c,\ s,\ t,\ b$, each transforming under the fundamental **3** representation of $SU(3)$. The different quark fields are called *flavours*. For each flavour, the three components of the **3** representation are called *colours*: $q = q_i$, $i = 1, 2, 3$. From the point of view of gauge invariance, there is nothing terribly fundamental about flavour, while the colour is an essential part of the underlying gauge symmetry structure. The terminology of flavour and colour is also frequently applied to other non-Abelian gauge theories.

Given what we know about non-Abelian gauge theories, we can write down the QCD Lagrangian immediately:

$$\mathscr{L} = -\frac{1}{4}(G^a_{\mu\nu})^2 + \sum_{q=u,d,c,s,t,b} \bar{q}(i\gamma^\mu D_\mu - m_q)q, \tag{66}$$

where $D_\mu = (\partial_\mu + ig_s t^a A_\mu)$ and m_q is the mass of quark q. That's it!

Even though this Lagrangian is simple, the dynamics it describes are anything but. When quantum loop corrections are included, the strength of the effective gauge coupling g_s becomes dependent on the energy of the process the theory is being used to describe. At high energies $E \gg$ GeV, the theory can be treated accurately in perturbation theory, with $g_s(E) \to 0$ as $E \to \infty$. This property is called *asymptotic freedom*. The flip side of asymptotic freedom is that the gauge coupling becomes large at lower energies, and perturbation theory becomes highly questionable to useless below $E \sim$ GeV. At energies below this, the underlying quark and gluon degrees of freedom undergo *confinement* in which they bind into colour-neutral bound states called *hadrons* such as pions and nucleons. This is an important illustration of how the fields in the underlying Lagrangian and the physical (particle) excitations of the theory need not line up with each other when the perturbative expansion in terms of couplings breaks down.

2 Symmetry Breaking and the Higgs Mechanism

Symmetries (and gauge invariances) play an extremely important role in physics. The existence of a symmetry constrains what is possible, and this often helps to make calculations easier. And through Noether's theorem, continuous symmetries lead to conservation laws. So far, we have encountered symmetries as transformations that act linearly on the fields of a theory and leave the Lagrangian invariant. It is also possible to have symmetries corresponding to non-linear field transformations. In particular, this emerges as part of a phenomenon called spontaneous symmetry breaking. When this is connected to gauge theories, one finds another important result called the Higgs mechanism.

2.1 Spontaneous Symmetry Breaking

In addition to being broken explicitly, symmetries can also be hidden. This occurs when the underlying action of a theory has a symmetry that is not respected by the vacuum state of the theory. It is often called spontaneous symmetry breaking (SSB), and it has very profound consequences in QFT. We will illustrate the process with a few examples, and then generalize.

***e.g.* SSB of a Discrete Symmetry**
Consider the Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)\ , \tag{67}$$

with the potential

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4 \ , \tag{68}$$

with μ^2 and λ both positive. The action clearly has a discrete symmetry under $\phi \to -\phi$. However, we also see that the quadratic term does not have the right sign to be a scalar mass term, unless we interpret the mass as $m = i\sqrt{\mu}$. Something is clearly wrong.

The way to resolve this can be found by looking at the shape of the potential, which we illustrate in Fig. 2. Evidently the origin of the field space, $\phi = 0$, is not a stable minimum of the potential. For example, if we were to start with the field at $\phi(t = t_0) = 0$ and $\partial_t\phi(t = t_0) \neq 0$, we would expect it to start rolling down the potential. Instead, the stable minima lie at

$$\langle\phi\rangle = \pm\mu/\sqrt{\lambda} \equiv \pm\, v \ . \tag{69}$$

The value of the field at the minimum is sometimes called the *vacuum expectation value*, or VEV for short. To proceed, we need to add a Step 0 to our recipe for dealing with quantum field theories in Sec. 1.1.3:

Recipe′:

0. Find the global minima of the potential. Choose one of them, and expand in fluctuations around it. The fluctuations should vanish in the vacuum configuration.
1. Start with the quadratic terms in the Lagrangian and extract from them the kinetic and mass terms.
2. For this, redefine the field variables to put the kinetic terms in *canonical* form and then diagonalize the mass matrices.
3. Add the terms higher than quadratic (in terms of the redefined and now-canonical/diagonal fields) and compute perturbatively with Feynman diagrams.

In this example, let us choose to expand around the minimum at $\langle\phi\rangle = +v$:

$$\phi(x) = v + h(x) \ , \tag{70}$$

where $h(x)$ is also a real scalar field. Plugging this form into the original Lagrangian, we see that the kinetic term for $h(x)$ is canonical while the potential becomes

$$\begin{aligned} V &= -\frac{1}{2}\mu^2(v+h)^2 + \frac{\lambda}{4}(v+h)^4 \\ &= -\frac{1}{4}\lambda v^4 + \frac{1}{2}(2\lambda\, v^2)\, h^2 + \lambda\, v\, h^3 + \frac{\lambda}{4}h^4 \ . \end{aligned} \tag{71}$$

This potential has a stable minimum at $h = 0$, a sensible mass term for h of $m_h = \sqrt{2\lambda}\, v$, and some h self-interactions.

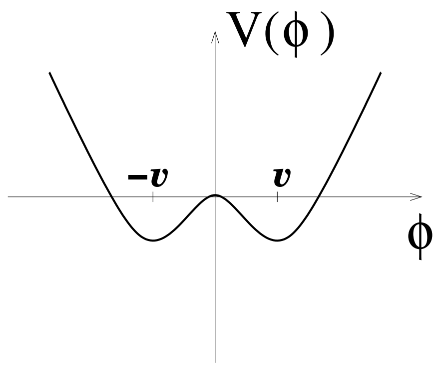

Figure 2: Scalar potential with spontaneous symmetry breaking.

The potential of Eq. (71) no longer has an obvious reflection symmetry, and is certainly not invariant under $h \to -h$. As a result, we say that the symmetry has been *spontaneously broken*. This is due to the fact that we expanded the theory around a particular vacuum state that does not get mapped back to itself by the symmetry. While this terminology is standard, it is also a bit of a misnomer because the Lagrangian still has a symmetry under

$$h(x) \ \to \ -2v - h(x) \ . \tag{72}$$

In contrast to the symmetries we studied before, this transformation acts non-linearly on h, since the transformed field is not a linear combination of the original fields due to the constant term. A more accurate description is that the symmetry has been hidden.

***e.g.* SSB of a Continuous Symmetry**
Things are even more interesting for continuous symmetries. Consider the globally $U(1)$-symmetric Lagrangian

$$\mathscr{L} = |\partial\phi|^2 - V(|\phi|) \tag{73}$$

with

$$V(|\phi|) = -\mu^2|\phi|^2 + \frac{\lambda}{2}|\phi|^4 \ . \tag{74}$$

This potential obviously has a global $U(1)$ symmetry (as does the kinetic term) and is sometimes called a "wine bottle" or a "Mexican hat". It looks just like that in Fig. 2 after rotating the profile around the vertical axis normal to the $Re(\phi)$-$Im(\phi)$ plane.

The global minima of the potential correspond to the base of the wine bottle, and are defined by the condition

$$|\phi|^2 = \mu^2/\lambda \ := \ v^2 \ . \tag{75}$$

Thus, the set of vacuum states is given by

$$\langle\phi\rangle = e^{i\beta}v \quad \leftrightarrow \quad |\beta\rangle. \tag{76}$$

Put another way, we have a circle's worth of distinct vacuum states that we can label by the parameter $\beta \in [0, 2\pi)$. These vacua do not have an energy barrier separating them, but there is an infinite gradient-energy cost to go from one state to another, and we must still pick a specific vacuum to expand around. Any such vacuum breaks the $U(1)$ invariance: $|\beta\rangle \to |\beta + \alpha\rangle$ under $\phi \to e^{i\alpha}\phi$.

Choosing the vacuum state $|\beta\rangle$ for some fixed value of β, we can expand about it by changing our field variables to a polar form:

$$\phi = (v + h(x)/\sqrt{2})e^{i(\beta+\rho(x)/\sqrt{2}v)} \ . \tag{77}$$

This form exchanges the two real degrees of freedom $\{Re(\phi),\, Im(\phi)\}$ for the polar variables $\{h(x),\, \rho(x)\}$. The advantage of the polar form is that both $h(x)$ and $\rho(x)$ vanish in the vacuum, and therefore represent fluctuations around it (as per Step 0). In general, you can choose any set of field variables you like as long as they lead to a sensible set of kinetic and mass terms, although a judicious choice can save you a lot of unneeded work. Plugging these new variables into the Lagrangian, we get

$$|\partial\phi|^2 = \frac{1}{2}(\partial h)^2 + \frac{1}{2}(1 + h/\sqrt{2}v)^2(\partial\rho)^2, \tag{78}$$

as well as

$$V(\phi) = (const) + \frac{1}{2}(2\lambda v^2)h^2 + \frac{\lambda}{\sqrt{2}}vh^3 + \frac{\lambda}{8}h^4. \tag{79}$$

This gives canonical kinetic terms for both h and ρ, some interactions, and masses of $m_h = \sqrt{2\lambda}\,v$ and $m_\rho = 0$. The Lagrangian in this form does not have an obvious rephasing symmetry, so again we say that the theory has undergone spontaneous symmetry breaking (SSB).

The masslessness of $\rho(x)$ here is not an accident. Under $U(1)$ transformations there is still a hidden symmetry under

$$\rho/\sqrt{2}v \to \rho/\sqrt{2}v + \alpha \ , \qquad h \to h \ . \tag{80}$$

In other words, the linear $U(1)$ has become a non-linear shift for ρ. This symmetry forbids non-derivative interactions involving ρ, and thus forbids a mass term for this field. It turns out that this is a generic feature of spontaneously broken continuous symmetries, and the corresponding massless states are called *Nambu-Goldstone Bosons* (NGBs).

Exercise: *Instead the polar form, try expanding this theory around the vacuum state* $|\beta = 0\rangle$ *using the variables* $\phi(x) = (v + \phi_1(x)/\sqrt{2}) + i\phi_2(x)/\sqrt{2}$. *Work out the kinetic terms and the particle masses, and figure out how the NGB emerges in terms of these variables.*

For general spontaneous symmetry breaking, we can relate the number of NGBs generated to the number of continuous symmetry generators broken [2, 4, 8]. Suppose the original continuous symmetry group G is spontaneously broken to a smaller subgroup H. Since it's continuous, elements of G can be written in the form $U(\alpha^a) = \exp(i\alpha^a t^a)$ for some generators t^a. The generators of G can split into a set of generators t^a_H of H plus the minimum number of other generators $t^a_{G/H}$ needed to make up a general G group element: $\{t^a\} = \{t^a_H\} \oplus \{t^a_{G/H}\}$. If $d(G)$ is the number of generators of G and $d(H)$ is the number for H, it can be shown that the number of NGBs is $d(G) - d(H)$. In other words, we get a NGB for every "broken" generator of G, corresponding to what is called the *coset space* G/H. For the example above, there was one symmetry generator for the global $U(1)$ and it was broken down to nothing leaving one NGB, as expected from the general argument.

2.2 SSB of Gauge Symmetries and the Higgs Mechanism

Having investigated the spontaneous breakdown of continuous global symmetries, it is natural to do the same for scalar theories with a gauge invariance. The simplest example has a single complex scalar and a $U(1)$ gauge invariance:

$$\mathscr{L} = |(\partial_\mu + igQA_\mu)\phi|^2 - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \ , \tag{81}$$

with the same potential as before,

$$V(\phi) = -\mu^2|\phi|^2 + \frac{\lambda}{2}|\phi|^4 \ . \tag{82}$$

The space of vacua is again $\langle\phi\rangle = ve^{i\beta}$. Choosing a fixed value of β, we expand around this vacuum by rewriting the complex scalar as

$$\phi(x) = e^{i(\beta+\rho(x)/\sqrt{2}v)}(v + h/\sqrt{2}) \ . \tag{83}$$

The main difference here compared to the global $U(1)$ theory studied earlier is that we now have the freedom to change the phase of ϕ by an amount that depends on spacetime. In particular, let us make a gauge transformation such that $\beta(x) + \rho(x)/\sqrt{2}v \to 0$ everywhere, or equivalently

$$\phi(x) \to (v + h(x)/\sqrt{2}) \ . \tag{84}$$

Since field configurations related by gauge transformations are physically equivalent, this choice of gauge will not affect our predictions for physical observables.

Expanding the theory in these new variables with this choice of gauge, we find

$$\begin{aligned} |D\phi|^2 &= |(\partial_\mu + igQA_\mu)\phi|^2 \\ &= \frac{1}{2}(\partial h)^2 + \frac{1}{2}\cdot 2\cdot g^2(v + h/\sqrt{2})^2 \, A_\mu A^\mu \ . \end{aligned} \tag{85}$$

This yields a nice kinetic term for $h(x)$, but also a mass term for the vector boson with value $m_A = \sqrt{2}gv$ along with some interactions between h and the vectors. However, thinking back to our previous discussion, a massless NGB mode seems to be missing.

To understand what has happened, let us compare the numbers of degrees of freedom (*dofs*) for $\langle\phi\rangle = 0$ and $\langle\phi\rangle \neq 0$. We have:

$$\langle\phi\rangle = 0: \quad \begin{cases} \phi & \textit{2 real dofs} \\ A_\mu \ \textit{(massless)} & \textit{2 independent polarizations} \end{cases}$$
$$\langle\phi\rangle \neq 0: \quad \begin{cases} \phi \to h & \textit{1 real dof} \\ A_\mu \ \textit{(massive)} & \textit{3 independent polarizations} \end{cases}$$

Aha! $2+2 = 1+3$, and the degrees of freedom match up in both cases. The would-be NGB mode of ϕ has gone to become the longitudinal polarization of the now-massive gauge boson. The highly technical term for this is that the NGB has been *eaten* by the gauge vector to give it mass. This effect is also called the *Higgs mechanism*, and the remaining physical scalar is called the *Higgs boson* of the theory [9, 10, 11, 12].[8]

3 The Standard Model, Finally

After all this preparation, we are now ready to tackle the Standard Model (SM). Well, almost ready; there's one more background thing we need to cover, but it will only take a minute. After that, it's all Standard Model all the time!

3.1 Chiral Fermions

Up to now we have been working with scalar, Dirac fermion, and vector fields. You may have wondered where these come from. The answer is that they are objects that transform under different representations of the Lorentz group. In analogy to the $U(1)$ and $SU(N)$ groups we discussed earlier, the Lorentz group is a continuous group that can be defined in terms of the commutation relations of a set of generators (called the Lie algebra). There is a total of six generators, and they can be identified with three boosts and three rotations. To build theories that are Lorentz invariant, we work with fields that transform under specific representations of the Lorentz group. The examples we've seen are:

$$\begin{aligned} x^\mu &\to x'^\mu = \Lambda^\mu{}_\nu x^\nu \\ \phi(x) &\to \phi'(x') = \phi(x) \\ \psi(x) &\to \psi'(x') = [M(\Lambda)]\psi(x) \\ A^\mu &\to A'^\mu = \Lambda^\mu{}_\nu A^\nu(x)\ . \end{aligned} \tag{86}$$

In terms of the language of group representations, we see that scalar fields transform under the trivial representation, spacetime coordinates and vector fields transform under the defining 4-vector representation of the group, and (Dirac) fermions transform under another representation of Lorentz. There are others, too, and they correspond to things like particles with even higher spins.

[8] Higgs was one of a number of people to discover this, with others including Anderson, Brout, Englert, Guralnik, Hagen, Nambu, and possibly a few others. Somehow it was Higgs' name that stuck.

Among these reps of Lorentz, let's focus more on the Dirac fermion. It can be shown that by choosing a nice basis for the fermions and the associated gamma matrices γ^μ, called the *chiral basis*, a general transformation matrix for a Dirac fermion can be written in the form [7]

$$M(\Lambda) \ = \ \left(\begin{array}{cc} e^{-i\alpha^i\sigma^i/2} & 0 \\ 0 & e^{-i\alpha^{i*}\sigma^i/2} \end{array}\right) \ , \tag{87}$$

where $\alpha^i = (\theta^i + i\beta^i)$, $i = 1, 2, 3$, are the group coordinates corresponding to the Lorentz transformation matrix Λ with rotation angles θ^i and boosts β^i. In contrast to the Lie groups we studied previously, the group coordinates are now complex and the corresponding representation matrices $M(\Lambda)$ are no longer unitary in general.

The key thing you should notice about Eq. (87) is that it is block diagonal. Suppose we write a 4-component Dirac fermion ψ in terms of two 2-component objects χ and $\bar{\xi}$,

$$\psi = \psi_L + \psi_R \ , \qquad \psi_L = \left(\begin{array}{c} \chi \\ 0 \end{array}\right) \ , \ \ \psi_R = \left(\begin{array}{c} 0 \\ \bar{\xi} \end{array}\right) \ . \tag{88}$$

The block form of the Lorentz transformation matrix on ψ then implies that the $\psi_L \sim \chi$ and $\psi_R \sim \bar{\xi}$ components transform independently from each other. These two 2-component fermions are said to transform in the left- and right-handed *Weyl* representations of the Lorentz group.

In this context, it's useful to rewrite the basic Dirac fermion Lagrangian in terms of these components. Doing this gives

$$\begin{aligned} \mathscr{L} \ &= \ \bar{\psi} i\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \\ &= \ \bar{\psi}_L i\gamma^\mu\partial_\mu\psi_L + \bar{\psi}_R i\gamma^\mu\partial_\mu\psi_R - m\left(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L\right) \ . \end{aligned} \tag{89}$$

We see that the kinetic terms for ψ_L and ψ_R are completely independent while the Dirac mass term mixes them with each other. Without the mass term, the two 2-component fields are completely disconnected from each other. This also links to the "L" and "R" parts of the names. With no mass, ψ_L describes a *left-handed* fermion particle with a spin that is anti-parallel to its momentum (and a right-handed anti-fermion), while ψ_R describes a *right-handed* fermion particle with spin that is parallel to its momentum (and a left-handed anti-fermion).[9] Adding a Dirac mass term mixes these chiralities with each other. In particular, with a mass term, there is now a rest frame for the particle: given a left-handed particle we could slow it down to rest and then reverse its direction to make it into a right-handed particle. This just isn't possible for a massless particle which does not have a rest frame.

If we wanted to, we could also write a Lorentz-invariant theory with only one of ψ_L or ψ_R. Let's do this for ψ_L:

$$\mathscr{L} = \bar{\psi}_L i\gamma^\mu\partial_\mu\psi_L \ . \tag{90}$$

This is a perfectly consistent theory that describes a massless left-handed fermion (and a right-handed anti-fermion). It turns out there is a way to write a Lorentz-invariant mass

[9] Note the chiral means "handed", as in left and right.

operator for ψ_L without adding any other fields. This is called a *Majorana* mass term and takes the form

$$\mathscr{L} \supset -M\,\psi_L^t(-i\gamma_2)\psi_L + (h.c.) \tag{91}$$

With such a mass, the interpretation is that ψ_L now describes a fermion of mass $|M|$ that is its own antiparticle and that can be left- or right-handed.

Exercise: *Given the Weyl fermions ψ_L and ψ_R transforming with charges Q_L and Q_R under a $U(1)$ symmetry (or gauge) group, what condition must the charges satisfy if a Dirac mass term involving is to be allowed by the symmetry? What is the condition on Q_L for a Majorana mass term for ψ_L to be allowed?*

3.2 The Symmetries, Fields, and Lagrangian of the SM

It's now easy to specify the Standard Model using the tools we've developed. The model is a non-Abelian gauge theory based on the group $SU(3)_c \times SU(2)_L \times U(1)_Y$. Of these factors, $SU(3)_c$ corresponds to the strong force, while $SU(2)_L \times U(1)_Y$ is called *electroweak* and is a combination of the weak and electromagnetic forces. Having fixed the underlying gauge group, all we need to do now is specify the matter content and the vacuum structure.

The fermionic matter in the SM comes in three copies called *families*. Each family consists of the following representations under $SU(3)_c \times SU(2)_L \times U(1)_Y$:

$$\begin{aligned}
Q_L &= (\mathbf{3}, \mathbf{2}, 1/6) = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \\
u_R &= (\mathbf{3}, \mathbf{1}, 2/3) \\
d_R &= (\mathbf{3}, \mathbf{1}, -1/3) \\
\\
L_L &= (\mathbf{1}, \mathbf{2}, -1/2) = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\
e_R &= (\mathbf{1}, \mathbf{1}, -1)
\end{aligned} \tag{92}$$

The numbers listed for $SU(3)_c$ and $SU(2)_L$ correspond to the dimension of the corresponding representation (with $\mathbf{1}$ meaning the trivial representation), while the $U(1)_Y$ number is the charge under this transformation. Note that we are specifying the representations for chiral fermions, and that the representations for the L and R fermions is different. The Q_L, u_R, and d_R fields transform non-trivially under $SU(3)_c$ and are called *quarks*, while the $SU(3)_c$-neutral L_L and e_R fields are called *leptons*. For the doublets under $SU(2)_L$, we have written out the two components as $Q_L = (u_L, d_L)^t$ and $L_L = (\nu_L, e_L)^t$. Each quark has three colour components, which we have not written explicitly.

Exercise: *Show that the gauge charges imply that it is impossible to write Dirac or Majorana mass terms for any of these fermions.*

In addition to three families of fermions, there is a single complex scalar Higgs field

$$H = (\mathbf{1}, \mathbf{2}, 1/2) = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} . \tag{93}$$

We will also write the gauge fields for the $SU(3)_c \times SU(2)_L \times U(1)_Y$ factors as

$$\begin{aligned} G^a_\mu &\sim (\mathbf{8}, \mathbf{1}, 0) \\ W^p_\mu &\sim (\mathbf{1}, \mathbf{3}, 0) \\ B_\mu &\sim (\mathbf{1}, \mathbf{1}, 0) \end{aligned} \tag{94}$$

Recall that the **8** of $SU(3)_c$ is the adjoint representation of the group, as is the **3** of $SU(2)_L$.

The notation we're using contains a lot of structure that is not shown explicitly. For example, each of u_L and d_L within Q_L is a fermion and a quark. The quark part means that each of u_L and d_L has three colour components, and each of these components itself is a fermion with further fermionic components. Don't worry if all this seems confusing, because most of the time you can get away without thinking about these details too much. However, if you ever get stuck, you can usually sort everything out by writing the various components explicitly. Let's do this for the $SU(3)_c \times SU(2)_L \times U(1)_Y$ components of the fermion ψ, which transforms according to

$$\begin{aligned} \psi_{ir} \to \psi'_{ir} &:= U^{(3)}_{ij} U^{(2)}_{rs} U^{(1)}\, \psi_{js} & (95) \\ &= (e^{i\alpha^a t^a_c})_{ij}\, (e^{i\beta^p t^p_L})_{rs}\, (e^{i\gamma Y})\, \psi_{js} \\ &= [\delta_{ij}\delta_{rs} + i\alpha^a (t^a_c)_{ij}\delta_{rs} + i\delta_{ij}\beta^p (t^p_L)_{rs} + i\delta_{ij}\delta_{rs}\gamma Y + \ldots]\, \psi_{js} . & (96) \end{aligned}$$

Here, i and j refer to the $SU(3)_c$ components and t^a_3 are the $SU(3)_c$ generators of the representation on ψ, r and s are the $SU(2)_L$ components and t^p_L are the $SU(2)_L$ generators, and Y is the $U(1)_Y$ charge. Each of these product subgroups acts independently relative to these indices. The quantities α^a, β^p, and γ are the group transformation parameters that apply universally to all representations. When a field transforms as a singlet under $SU(3)_c$ or $SU(2)_L$, the corresponding representation generators vanish and we do not need to include an index for that group on the field. Explicitly,

$$Q_L = (Q_L)_{ir}, \quad u_R = (u_R)_i, \quad d_R = (d_R)_i\ , \quad L_L = (L_L)_r, \quad e_R = (e_R)\ . \tag{97}$$

Woohoo!

With the gauge structure and fields in place, most of the SM Lagrangian is already fixed. To keep things simple(r), for now I'll write it for only one generation of SM fermions. It is

$$\mathscr{L} = \mathscr{L}_{gauge} + \mathscr{L}_{Higgs} + \mathscr{L}_{Yukawa}\ . \tag{98}$$

The gauge piece is completely fixed by gauge invariance:

$$\begin{aligned} \mathscr{L}_{gauge} &= -\frac{1}{4}(G^a_{\mu\nu})^2 - \frac{1}{4}(W^p_{\mu\nu})^2 - \frac{1}{4}(B_{\mu\nu})^2 \\ &\quad + \overline{Q}_L i\gamma^\mu D_\mu Q_L + \overline{u}_R i\gamma^\mu D_\mu u_R + \overline{d}_R i\gamma^\mu D_\mu d_R \\ &\quad + \overline{L}_L i\gamma^\mu D_\mu L_L + \overline{e}_R i\gamma^\mu D_\mu e_R\ , \end{aligned} \tag{99}$$

where each covariant derivative takes the form

$$D_\mu = \partial_\mu + ig_s\, t_c^a G_\mu^a + ig\, t_L^p W_\mu^p + ig' Y B_\mu, \tag{100}$$

with t_c^a the appropriate $SU(3)_c$ generators for the coresponding rep ($t_c^a = 0$ for the trivial rep), t_L^p the generators for $SU(2)_L$ ($t_L^p = 0$ for the trivial rep), and Y is the charge of the field under $U(1)_Y$ and is called *hypercharge*. The Higgs part is

$$\mathscr{L}_{Higgs} = \left|\left(\partial_\mu + ig\,\frac{\sigma^p}{2}W_\mu^p + ig'\frac{1}{2}B_\mu\right) H\right|^2 - \left(-\mu^2|H|^2 + \frac{\lambda}{2}|H|^4\right)\ . \tag{101}$$

This potential induces spontaneous symmetry breaking. Finally, the third set of terms in the SM Lagrangian correspond to scalar-fermion *Yukawa* interactions,

$$\mathscr{L}_{Yukawa} = -y_u \overline{Q}_L \tilde{H} u_R - y_d \overline{Q}_L H d_R - y_e \overline{L}_L H e_R + (h.c.)\ , \tag{102}$$

where $\tilde{H} := i\sigma^2\Phi = (H^{0*},\, -H^{+*})^t$. These interactions are the most general ones we can write (at the renormalizable level) that are consistent with gauge invariance given the charges of Eq. (92). Note that the gauge charges forbid fermion mass terms at this stage.

3.3 Higgs, Electroweak Symmetry Breaking, and Mass

The first step in working out the implications of this Lagrangian is to determine the vacuum structure. The Higgs potential leads to spontaneous symmetry breaking. To study this, it is conveninent to choose a gauge (called the *unitarity gauge*) such that

$$H(x) = \begin{pmatrix} 0 \\ v + h(x)/\sqrt{2} \end{pmatrix}\ , \tag{103}$$

where $v = \sqrt{\mu^2/\lambda}$ is the Higgs vacuum expectation value (VEV), and the real scalar h field is called the Higgs boson. This VEV has hugely important consequences for the rest of the theory.

With the Higgs vacuum structure of Eq. (103), it is helpful to figure which parts of the SM gauge group are broken and which are left invariant. Since $H(x)$ does not transform under $SU(3)_c$, the Higgs VEV has no effect on colour. With a bit of fiddling, one can also show that the maximal unbroken subgroup of $SU(2)_L \times U(1)_Y$ is the $U(1)$ generated by

$$Q \ \equiv\ t_L^3 + Y\ . \tag{104}$$

We identify this unbroken electroweak subgroup with the $U(1)_{em}$ invariance of electromagnetism. The Q generator defined here corresponds to electric charge, and we expect a massless vector boson to go with it.

Exercise: *Show that the $U(1)_{em}$ subgroup of $SU(2)_L \times U(1)_Y$ acting on $H(x)$ in the unitary gauge leaves it invariant. Also, work out the values of Q for u_L, d_L, u_R, d_R, e_L, and e_R.*

Based on the Higgs toy model we studied in Sec. 2.2, we expect that some of the electroweak gauge bosons will get masses. To see this in detail, we turn to the covariant kinetic term for the Higgs field:

$$|D_\mu H|^2 \to \frac{1}{2}(\partial h)^2 + \frac{1}{2}\frac{v^2}{2}\left[g^2[(W_\mu^1)^2 + (W_\mu^2)^2] + (-gW_\mu^3 + g'B_\mu)^2\right] + \dots \tag{105}$$

From this expression it is clear that two orthogonal linear combinations of W_μ^1 and W_μ^2 obtain equal masses. It is convenient to choose the two combinations that have definite charges $Q = \pm 1$ under electromagnetism,

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp iW_\mu^2\right) \ . \tag{106}$$

We identify them with the $W^\pm$ vector bosons of the weak interaction, and they have mass

$$m_W^2 = \frac{g^2}{2}v^2 \ . \tag{107}$$

For W_μ^3 and B_μ we get a squared mass matrix of

$$M^2 = \frac{v^2}{2}\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} \ . \tag{108}$$

As expected, this matrix has a zero eigenvalue corresponding to the photon γ (and field A_μ). The other linear combination of W_μ^3 and B_μ is called the neutral Z^0 vector boson (with field Z^0). These mass eigenstates are related to the fields in the original basis by

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} \ , \tag{109}$$

where the *weak mixing angle* θ_W is defined by

$$\sin\theta_W = s_W = \frac{g'}{\sqrt{g^2 + g'^2}} \ , \qquad \cos\theta_W = c_W = \frac{g}{\sqrt{g^2 + g'^2}} \ . \tag{110}$$

While the photon is massless, the Z^0 vector boson has mass

$$m_Z^2 = \left(\frac{g^2 + g'^2}{2}\right)v^2 \ . \tag{111}$$

The longitudinal components of the massive $W^\pm$ and Z^0 vectors account for the missing NGBs from the three broken electroweak generators. Since the new mass eigenstate vector fields we have defined above are related to the original gauge eigenstates by orthogonal transformations, the kinetic terms for the mass eigenstate vectors are also canonical.

Interactions between these electroweak vector boson mass eigenstates and the fermions of the SM are dictated by the gauge-covariant derivatives. These take the form (with $t^p = 0$

for $SU(2)_L$ singlets)

$$
\begin{aligned}
D_\mu &\supset igt^pW^p_\mu + ig'YB_\mu \\
&= ig\left[\frac{1}{\sqrt{2}}(t^1+it^2)W^+_\mu + \frac{1}{\sqrt{2}}(t^1-it^2)W^-_\mu\right] \\
&\qquad + i(gc_Wt^3 - s_Wg'Y)Z_\mu + i(gs_Wt^3+g'c_WY)A_\mu \\
&= ig\left[\frac{1}{\sqrt{2}}(t^1+it^2)W^+_\mu + \frac{1}{\sqrt{2}}(t^1-it^2)W^-_\mu\right] + i\bar{g}(t^3-s_W^2Q)\,Z_\mu + ie\,Q\,A_\mu\ .
\end{aligned}
\tag{112}
$$

In the last line, we have implicitly defined the couplings

$$
e = \frac{gg'}{\sqrt{g^2+g'^2}} = gs_W = g'c_W\ , \qquad\qquad \bar{g} = \sqrt{g^2+g'^2}\ . \tag{113}
$$

While the SM has many individual vector boson interaction terms, we see that they are all essentially fixed by the values of g, g', and v from the underlying gauge-invariant theory. Measurements of the electroweak sector of the SM find that [13]

$$
\begin{aligned}
m_W &\simeq 80.4\ \text{GeV}, \quad m_Z \simeq 91.2\ \text{GeV}, \quad v \simeq 174\ \text{GeV}, \\
s_W^2 &\simeq 0.23, \quad g \simeq 0.65, \quad g' \simeq 0.35, \quad e^2/4\pi \simeq 1/137.
\end{aligned}
\tag{114}
$$

Note that not all the values of these measurable masses and couplings are independent in the underlying theory. We will see that this allows for very stringent experimental tests of the SM.

Exercise: *Work out the couplings of the u_L and d_L parts of the Q_L doublet to the $W^\pm_\mu$, Z_μ, and A_μ vector bosons. Do the same for u_R and d_R.*

The remaining parts of the SM Lagrangian to deal with are the Yukawa terms. Rewriting the Higgs scalar doublet in terms the new vacuum-friendly field variables, the Yukawa interactions become

$$
\begin{aligned}
-\mathscr{L}_{Yukawa} &= y_u\overline{Q}_L\tilde{H}\,u_R + y_d\overline{Q}_LH\,d_R + y_e\overline{L}_LH\,e_R + (h.c.) \qquad (115)\\
&= y_u\,(v+h/\sqrt{2})\,\overline{u}_Lu_R + y_d\,(v+h/\sqrt{2})\,\overline{d}_Ld_R + y_e\,(v+h/\sqrt{2})\,\overline{e}_Le_R + (h.c.)\ .
\end{aligned}
$$

This expression consists of Dirac mass terms for the fermions together with fermion-Higgs boson interactions:

$$
m_i = y_i\,v\ . \tag{116}
$$

In other words, the mass of each SM fermion is proportional to how strongly it couples to the Higgs field.

The Feynman rules for the SM are summarized in Appendix B.

3.4 Flavour in the Standard Model

In our discussion above, we did not say anything about the three families of quarks and leptons to simplify the discussion. There is a very interesting story here, and we turn to it now. The SM contains three copies of all the fermion representations that we call *families*, *generations*, or *flavours*. They are (in order of increasing mass)

$$\text{Quarks :} \quad \begin{cases} u_{L,R} & c_{L.R} & t_{L,R} & Q = +2/3 \\ d_{L,R} & s_{L,R} & b_{L,R} & Q = -1/3 \end{cases} \tag{117}$$

$$\text{Leptons :} \quad \begin{cases} \nu_{e_L} & \nu_{\mu_L} & \nu_{\tau_L} & Q = \quad 0 \\ e_{L,R} & \mu_{L,R} & \tau_{L,R} & Q + -1 \end{cases} \tag{118}$$

The first column corresponds to the first generation, the second column to the second generation, and the third column to the third. The elements of each generation have identical sets of $SU(3)_c \times SU(2)_L \times U(1)_Y$ quantum numbers (*i.e.* representations) but differ greatly in their masses.

Instead of writing out all three generations explicitly, it is much easier to use a condensed notation with a generation index $A = 1, 2, 3$. For example, we will write u_{R_A} with

$$u_{R_{A=1}} = u_R \ , \quad u_{R_2} = c_R \ , \quad u_{R_3} = t_R \ , \tag{119}$$

and similarly for the other states. Since all three generations have identical quantum numbers, we can choose our field variables such that all the gauge-covariantized kinetic terms are diagonal in generation space and canonically normalized. That is

$$\mathscr{L}_{gauge} \supset \overline{Q}_{L_A} i\gamma^\mu D_\mu Q_{L_A} + \overline{u}_{R_A} i\gamma^\mu D_\mu u_{R_A} + \ldots \tag{120}$$

This choice of field variables is sometimes called the *gauge eigenbasis*. We will always start off with this basis and work from there.

Going back to the Yukawa interactions, gauge invariance allows them to have a non-trivial family-mixing structure. The most general set of gauge-invariant Yukawa terms is

$$\begin{aligned} -\mathscr{L}_{Yukawa} &= y_{u_{AB}} \overline{Q}_{L_A} \tilde{H}\, u_{R_B} + y_{d_{AB}} \overline{Q}_{L_A} H\, d_{R_B} + y_{e_{AB}} \overline{L}_{L_A} H\, e_{R_B} + (h.c.) \qquad (121) \\ &= (v + h/\sqrt{2}) \overline{u}_{L_A} y_{u_{AB}} u_{R_B} + (v + h/\sqrt{2}) \overline{d}_{L_A} y_{d_{AB}} d_{R_B} + (v + h/\sqrt{2}) \overline{e}_{L_A} y_{e_{AB}} e_{R_B} + (h.c.) \\ &= (v + h/\sqrt{2})\, \overline{u}_L y_u u_R + (v + h/\sqrt{2})\, \overline{d}_L y_d d_R + (v + h/\sqrt{2})\, \overline{e}_L y_e e_R + (h.c.) \end{aligned}$$

In the third line, we have implicitly contracted the generation indices to write this expression in terms of matrices and row and column vectors in generation space. Compared the one-generation case, the Yukawa couplings have now become complex-valued 3×3 matrices, and the fermion mass terms are mass matrices.

To do perturbation theory with this more general Lagrangian, we want to diagonalize the mass matrices and identify the mass eigenstates which correspond to specific particles. The trick to this is to use the fact that any complex matrix can be bi-diagonalized by a pair of unitary matrices. To achieve this, define a new set of rotated fields according to

$$
\begin{aligned}
u_{L_A} &= V^L_{u_{AB}}\, u'_{L_B}\ , \qquad u_{R_A} = V^R_{u_{AB}}\, u'_{R_B}\ ,\\
d_{L_A} &= V^L_{d_{AB}}\, d'_{L_B}\ , \qquad d_{R_A} = V^R_{d_{AB}}\, d'_{R_B}\ ,\\
\\
e_{L_A} &= V^L_{e_{AB}}\, e'_{L_B}\ , \qquad e_{R_A} = V^R_{e_{AB}}\, e'_{R_B}\ ,\\
\nu_{L_A} &= V^L_{\nu_{AB}}\, \nu'_{L_B}\ ,
\end{aligned}
\tag{122}
$$

where the $V_f^{L,R}$ are unitary matrices in generation space. We can choose them such that they bi-diagonalize the Yukawa interaction matrices. That is

$$
\begin{aligned}
V_u^{L\dagger} y_u V_u^R &= \frac{1}{v}\, diag(m_u, m_c, m_t)\\
V_d^{L\dagger} y_d V_d^R &= \frac{1}{v}\, diag(m_d, m_s, m_b)\\
V_e^{L\dagger} y_e V_e^R &= \frac{1}{v}\, diag(m_e, m_\mu, m_\tau)
\end{aligned}
\tag{123}
$$

In terms of the primed fields, the Yukawa interactions containing the mass terms are now diagonal. For example

$$
\begin{aligned}
-\mathscr{L}_{Yukawa} &\supset (v + h/\sqrt{2})\, u_L y_u u_R\\
&= (v + h/\sqrt{2})\, \overline{u}'_L\, (V_u^{L^\dagger} y_u V_u^R)\, u'_R\\
&= (1 + h/\sqrt{2}v)\, \Big(m_u\, \overline{u}'_L u'_R + m_c\, \overline{c}'_L c'_R + m_t\, \overline{t}'_L t'_R\Big)\ .
\end{aligned}
\tag{124}
$$

Since these field transformations are unitary (and global), the fermion kinetic terms retain their generation-diagonal canonical form. For instance,

$$
\begin{aligned}
\overline{Q}_L i\gamma^\mu \partial_\mu Q_L &\to \overline{u}'_L V_u^{L\dagger} i\gamma^\mu \partial_\mu V_u^L u'_L + \overline{d}'_L V_d^{L\dagger} i\gamma^\mu \partial_\mu V_d^L d'_L\\
&= \overline{u}'_L i\gamma^\mu \partial_\mu u'_L + \overline{d}'_L i\gamma^\mu \partial_\mu d'_L\ .
\end{aligned}
\tag{125}
$$

These keep the same form because the kinetic terms only have LL and RR pieces, and do not mix the upper and lower components of the $SU(2)_L$ doublets. As a result, we always get the combination $V_f^{L,R^\dagger} V_f^{L,R} = \mathbb{I}$ in generation space. The primed field basis we have defined thus has canonical kinetic terms and diagonal masses, and is therefore a good basis for perturbation theory. It is often called the *mass eigenbasis*.

Let us turn next to the effects of diagonalizing the fermion mass matrices on the rest of the theory. By construction, or from Eq. (124), we see that the couplings of the primed fields to the Higgs boson h are all generation-diagonal. The couplings of the fermions to the photon A_μ, the massive Z_μ vector, and the gluon G^a_μ are also diagonal in generation space. This comes about for exactly the same reason that the fermion kinetic terms remain diagonal – the unitary transformations cancel each other out.

Things are more interesting for the couplings of fermions to the massive $W^\pm_\mu$ vectors. Here we have

$$\begin{aligned} -\mathscr{L} &\supset \frac{g}{\sqrt{2}}\,\overline{u}_L\gamma^\mu W^+_\mu d_R + \frac{g}{\sqrt{2}}\,\overline{\nu}_L\gamma^\mu W^+_\mu e_R + (h.c.) \\ &= \frac{g}{\sqrt{2}}\,\overline{u}'_L(V_u^{L\dagger}V_d^L)\gamma^\mu W^+_\mu d'_R + \frac{g}{\sqrt{2}}\,\overline{\nu}'_L(V_\nu^{L\dagger}V_e^L)\gamma^\mu W^+_\mu e'_R + (h.c.)\ . \end{aligned} \tag{126}$$

The unitary generation-space matrix appearing in the quark term is called the Cabibbo-Kobayashi-Maskawa (CKM) matrix,

$$V^{(CKM)} = V_u^{L\dagger}V_d^L = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}\ . \tag{127}$$

This represents a physical cross-generational mixing. For the leptons, on the other hand, we can always choose the neutrino mixing matrix $V_\nu^L = V_e^L$ without changing anything else in the SM Lagrangian.[10] Thus, we can set our field variables such that the couplings of the W to leptons remain generation-diagonal. The only genuine source of flavour mixing in the SM is therefore the CKM matrix.

Generation mixing is observed experimentally and seems to be consistent with the CKM picture. Numerically, the magnitudes of the entries in the CKM matrix are [13]

$$|V^{(CKM)}| \simeq \begin{pmatrix} 0.9748 & 0.226 & 0.0041 \\ 0.220 & 0.995 & 0.042 \\ 0.0082 & 0.040 & 1.0 \end{pmatrix}\ . \tag{128}$$

The number of decimal places here corresponds to the current experimental precision.

The Yukawa couplings we began with (in the gauge eigenbasis) in Eq. (121) can be complex. This leads to complex phases in the CKM matrix. In general, one can write a 3×3 unitary matrix in terms of three rotation angles ($O(3)\subset SU(3)$) and six phases. Five of these phases can be removed by field redefinitions that leave the real, diagonal form of the mass and kinetic terms unchanged. The remaining phase is physical, and gives rise to observable CP violation.

3.5 Testing the Electroweak Sector of the SM

With the SM theory in hand, we can now make theoretical predictions and compare them to data. To do this, we first have to use a subset of data to fix the parameters of the SM Lagrangian. Once we've done this, we can calculate "output" observables in terms of the "input" observables. I'll illustrate this here for the electroweak sector of the SM. See Refs. [13, 14, 15] for more detail.

In this sector of the SM, the three parameters that we need to fix are the gauge couplings g and g', and the Higgs VEV v. However, in practice it is more convenient to use inputs that are closer to observables and that are measured well. A nice set is α, G_F, and m_Z.

[10] This would not be true if it were possible to write a mass term for the neutrinos in the SM.

The electromagnetic coupling α_{em} is determined at low energy from the anomalous magnetic moment of the electron, and is then extrapolated up to a value relevant for physics at energies close to m_Z. The current status is [13, 16]

$$\begin{aligned} \alpha(m_Z) &:= \frac{e^2}{4\pi} = \frac{g^2 s_W^2}{4\pi} \\ &= (127.95 \pm 0.02)^{-1} . \end{aligned} \tag{129}$$

Muon decays are used to extract the *Fermi constant* G_F, which is given by

$$\begin{aligned} G_F &:= \frac{1}{2\sqrt{2}\, v^2} \\ &= (1.1663787 \pm 0.00000006) \times 10^{-5} \text{ GeV}^{-2} . \end{aligned} \tag{130}$$

The mass of the Z^0 is deduced from the energy dependence of the $e^+e^- \to f\bar{f}$ cross section for $\sqrt{s} \sim m_Z$:

$$\begin{aligned} m_Z &= \frac{e}{\sqrt{2}\, s_W c_W}\, v \\ &= (91.188 \pm 0.002) \text{ GeV} . \end{aligned} \tag{131}$$

It is straightforward to solve for e, s_W, and v from these expressions:

$$e = \sqrt{4\pi\,\alpha} \simeq 0.313 \tag{132}$$

$$v = 1/\sqrt{2\sqrt{2}G_F} \simeq 174 \text{ GeV} \tag{133}$$

$$s_W^2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\pi\alpha/\sqrt{2}G_F m_Z^2} \simeq 0.234 . \tag{134}$$

Note that we take s_W to be the positive square root, with c_W positive as well.

Having fixed the input values of the Lagrangian, we can now go on to compute any other electroweak observable we would like. The most useful of these for testing the SM are usually the so-called *Z-pole* observables, corresponding to processes of the form $e^+e^- \to f\bar{f}$ at centre-of-mass (CM) energies near the Z^0 mass, $s = (p_{e^-} + p_{e^+})^2 \simeq m_Z^2$. In this regime, the dominant contribution to the cross section comes from the diagram with a Z^0 in the s-channel. The dominance of this diagram comes about because of the form of the Z^0 propagator denominator appearing in the scattering amplitude:

$$\mathcal{M} \propto \frac{1}{p^2 - m_Z^2 - i m_z \Gamma_Z} , \tag{135}$$

where $\Gamma_Z = 1/\tau_Z$ is the total decay rate of the Z^0. Since $\Gamma_Z \ll m_Z$, this becomes very large when $p^2 = s = m_Z^2$. Precision measurements of the energy dependence of the cross section for $e^+e^- \to f\bar{f}$ as a function of $\sqrt{s}$ at the LEP-I (CERN) and SLC (SLAC) colliders find a clear mass peak for the Z^0, shown in Fig. 3. Based on the location and shape of the peak, it is possible to extract m_Z and Γ_Z. The value of m_Z is used as an input observable, but [13]

$$\Gamma_Z = (2.495 \pm 0.002) \text{ GeV} \tag{136}$$

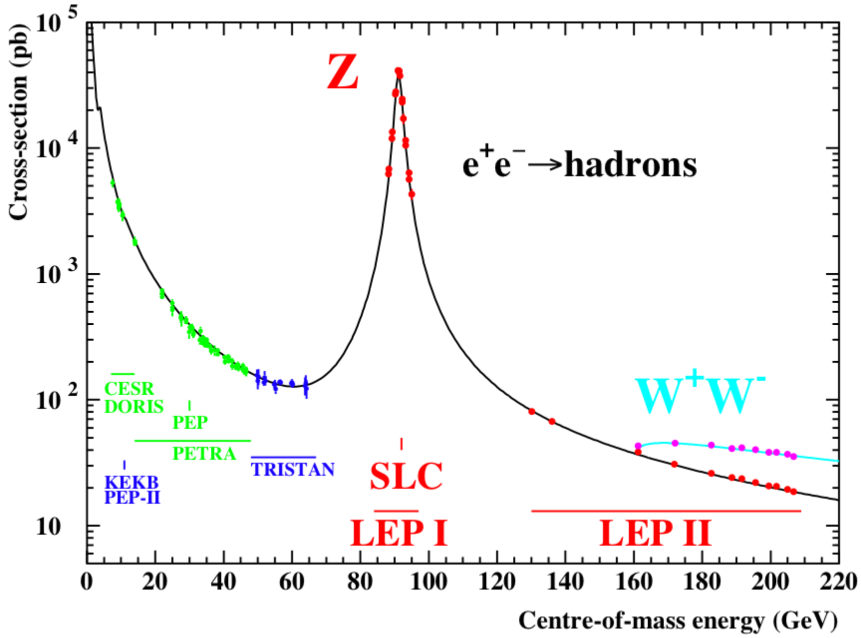

Figure 3: Cross section for $e^+e^- \to$ hadrons (and W^+W^-) at various energies.

is an independent output observable whose value we can also be computed in terms of the inputs. Since the decay rate Γ_Z gets contributions from "invisible" decays to neutrinos, we can compare the observed value to the SM prediction to figure out how many (active) neutrinos there are. It turns out that data and SM predictions only agree with each other for three neutrino generations.

Besides the decay rate, there are many other useful Z-pole observables such as the relative production rates of hadrons and leptons, the relative polarization fractions of the particles produced, and the probability for a fermion to be produced along the direction of the e^- beam relative to it being produced in the direction of the e^+ beam (in an e^+e^- collision in the centre-of-mass frame). A summary of these observables compared to SM predictions based on a global fit to data is shown in left panel of Fig. 4, which is based on the analysis of Ref. [17].

In addition to these primarily Z-pole observables, there are also some very good tests of the electroweak structure of the SM at both lower and higher energies. At lower energies, precision measurements neutrino cross-sections, atomic parity violation, and the determination of the τ lifetime are especially important. Higher energy colliders such as the Tevatron and the LHC have measured m_W very precisely [19, 20]:

$$m_W = (80.387 \pm 0.019)\ \text{GeV} \tag{137}$$

These colliders have also measured rates of single and double W and Z production [18]. Comparisons of SM predictions to data for these and other production cross sections are shown in the right panel of Fig. 4.

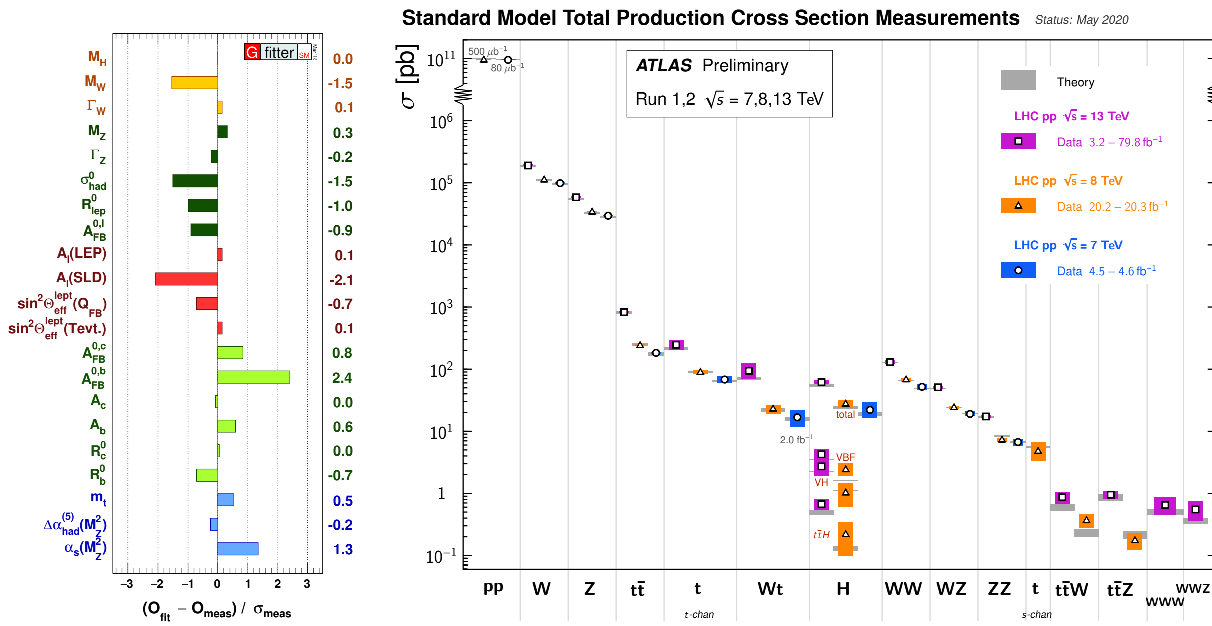

Figure 4: *Left:* Observed values of various electroweak observables compared to the best-fit predictions of the SM, from Ref. [17]. *Right:* Production cross sections for the electroweak vector bosons measured at the LHC by the ATLAS experiment compared to theory [18].

3.6 Testing the SM Higgs

The last particle predicted by the SM to be found was the Higgs boson. The discovery of a new particle of mass $m \simeq 125$ GeV with the right properties to be the SM Higgs was announced on July 4, 2012.[11] In the years since, many further measurements have confirmed that this new particle has the properties expected of the SM Higgs, giving us confidence that it is the genuine article. In this section, we discuss how the SM Higgs boson decays, how it can be produced in high-energy collisions, and how these properties were used to discover it. More detailed reviews can be found in Refs. [21, 22, 23, 24].

Producing the Higgs boson and detecting is enormously difficult. This particle is unstable, and decays nearly immediately after it is created. This means there aren't any Higgses just sitting around so we have to make them ourselves. Doing so requires high-energy particle colliders such as the LHC. In the case of the LHC, which collides beams of protons at a CM energy near $E_{CM} \simeq 13$ TeV, it turns out that you only get about one Higgs boson for every 20 billion proton collisions. And just about as soon as you get one, it decays back to other SM particles that you need to identify in your detectors.

The largest production channel for the Higgs at proton colliders like the LHC is *gluon fusion*, in which a pair of gluons coming from the colliding protons fuse to make a Higgs. While the Higgs doesn't couple directly to gluons, it connects indirectly through loop diagrams involving (primarily) the top quark as shown in the top diagram of the left panel

[11] *Higgsdependence Day!*

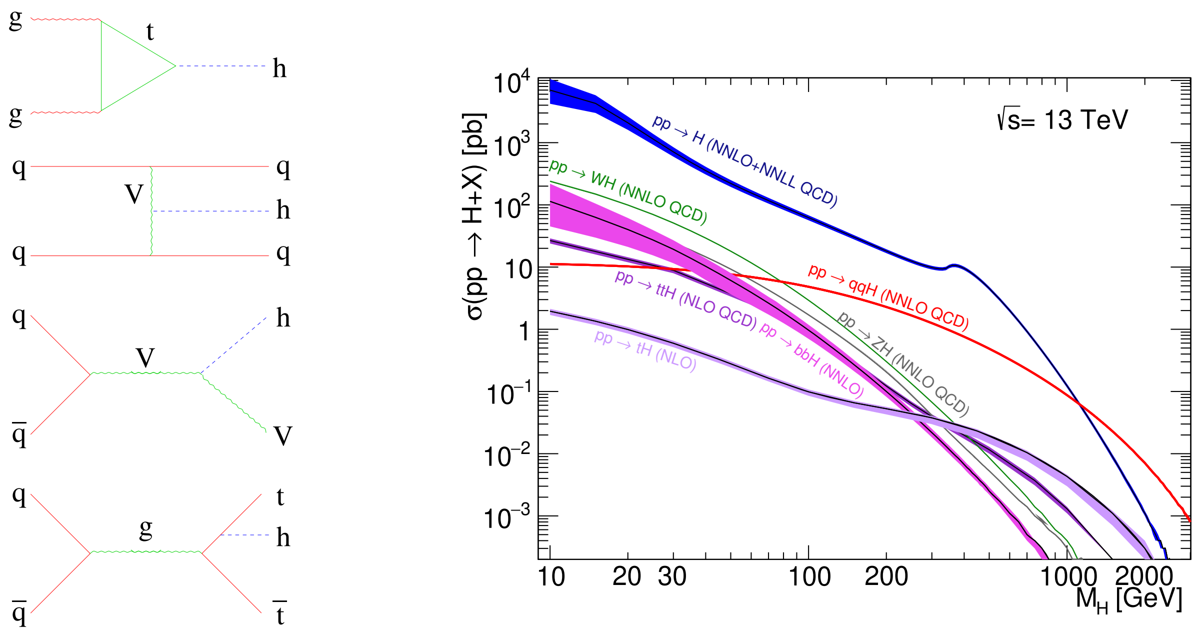

Figure 5: *Left:* Diagrams contributing to Higgs production at a hadron collider through gluon fusion, vector boson fusion, Vh associated production, and $t\bar{t}h$. *Right:* Production cross sections for these modes at the LHC with $\sqrt{s} = 13$ TeV.

of Fig. 5. Other important production channels include vector boson fusion (VBF) in which a pair of W or Z bosons join to make a Higgs (second diagram on the left in Fig. 5), vector-boson associated production Vh with $V = W, Z$ (third diagram), and $t\bar{t}h$ associated production (fourth diagram). The right panel of Fig 5 shows the cross sections of these processes in proton collisions at $E_{CM} = \sqrt{s} = 13$ TeV as a function of the Higgs mass. Note that the observed mass value is $m_h = 125$ GeV. In $e + e^-$ colliders running at lower CM collision energies $E_{CM} = \sqrt{s} \sim 250$ GeV, such as in proposed "Higgs Factory" colliders, the dominant production channel is Zh associated production [25], also called *higgstrahlung* in this context.

To identify and test the Higgs after it has been produced, we need to understand how it decays. Recall that the Higgs boson couples to other particles in the SM proportionally to their mass. This implies that it decays most often to the heaviest particles it can. For a SM Higgs with mass $m_h = 125$ GeV, the leading decay mode is $h \to b\bar{b}$. After this, the most important decays involve WW^*, gg, ZZ^*, $\tau\bar{\tau}$, and $c\bar{c}$.[12] The relative probabilities for the Higgs to decays to the various channels are called *branching ratios* (BR). They are given by

$$\mathrm{BR}(h \to f) = \Gamma_f / \Gamma_{tot} \ , \tag{138}$$

where Γ_f is the decay rate for the Higgs to decay to the final state f and $\Gamma_{tot} = \sum_f \Gamma_f$ is the total decay rate. In the right panel of Fig. 6 we show the SM predicted Higgs branching ratios as a function of the Higgs mass (with the observed value being $m_h = 125$ GeV).

[12] Note that $h \to WW^*$, ZZ^* here refers to Higgs decays to one on-shell and one off-shell vector boson, where the off-shell vector only appears as an intermediate virtual particle.

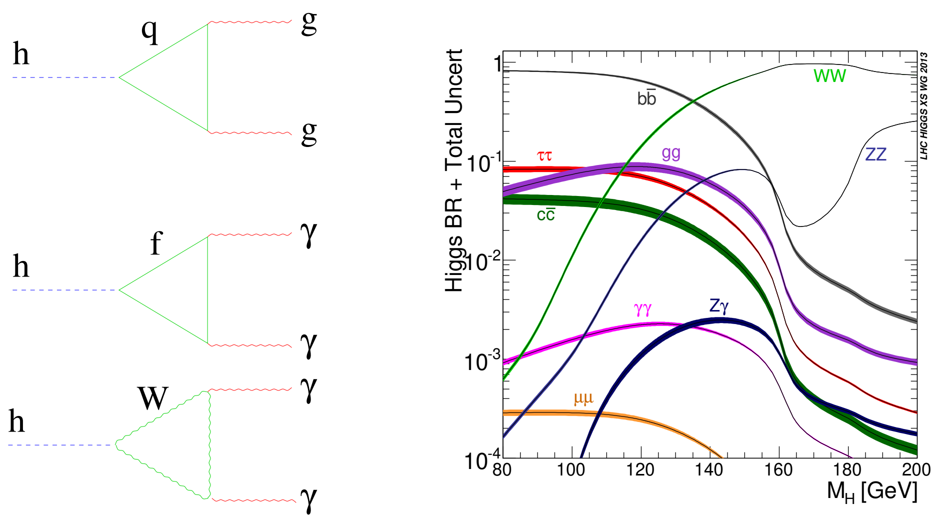

Figure 6: *Left:* Dominant loop diagrams contributing to $h \to gg$ and $h \to \gamma\gamma$. *Right:* Branching fractions of the SM Higgs boson as a function of the Higgs mass for the most important decay channels.

So far, the Higgs has only been measured conclusively at the LHC. Even before its discovery, precision electroweak measurements gave strong indirect evidence for the existence of a Higgs with mass $m_h \lesssim 160$ GeV [17], while unsuccessful direct searches at LEP II in e^+e^- collisions at $\sqrt{s} \simeq 208$ GeV imposed a lower mass bound of $m_h \gtrsim 114$ GeV [26]. Finding the Higgs took a huge amount of data at the much higher (proton) collision energies of the LHC. The most important discovery channel was through the Higgs decay channel $h \to \gamma\gamma$. In analogy to $gg \to h$, this decay is generated by loop effects with top quarks or W bosons, as shown in the lower two diagrams in the left panel of Fig. 6. Even the branching ratio for this channel is small, $BR(h \to \gamma\gamma) \simeq 0.002$, the two-photon final is relatively easy to identify in the LHC detectors. These decays can also be identified by combining the measured 4-momenta $p_{1,2}$ of the photons into an *invariant mass*,

$$m_{\gamma\gamma} = \sqrt{(p_1 + p_2)^2} \ . \tag{139}$$

Applying energy-momentum conservation, it is not hard to check that $m_{\gamma\gamma} \simeq m_h$ for a pair of photons from a decaying Higgs boson. In contrast, the diphoton invariant mass distribution from background events falls smoothly, and both features were seen in the data [27]. A second important contribution to the discovery was obtained from the $h \to ZZ^*$ decay channel, with each Z decaying to a pair of leptons. In this case, the 4-lepton invariant mass is expected to reconstruct the Higgs mass.

Exercise: *Prove the invariant mass relations above for $h \to \gamma\gamma$.*

Following the initial discovery, the Higgs boson has been studied in a number of other production channels and decay modes. So far, all the data is consistent with the predictions of the SM [27, 28]. Making a set of fairly mild assumptions (that are consistent with a SM

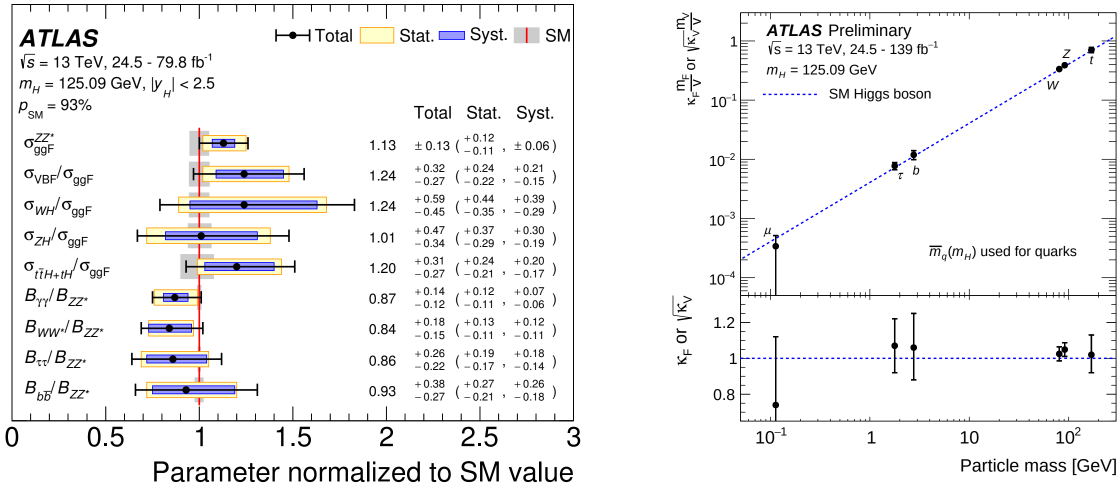

Figure 7: *Left:* Higgs production rates measured by the ATLAS experiment at the LHC in a number of production channels. *Right:* Higgs couplings to other SM particles derived from the data and plotted in relation to the particle masses. Both figures are from Ref. [27].

Higgs), the production rates and couplings of the Higgs to many other SM particles can be extracted from the experimental data [27, 28]. The production rates in various channels match the predictions of the SM, and couplings follow the particle masses as expected for the SM. Results from the ATLAS experiment at the LHC exhibiting these features are shown in Fig. 7 [27]. Despite these successes of the SM Higgs description, a great deal of further study at the LHC and future colliders is needed to test the Higgs sector thoroughly.

3.7 Testing the Strong Force

One question you may still have about the SM is how we get the observed hadronic particles like protons and mesons out of it. Related to this is the puzzle of why we never see free quarks on their own. The answer to both questions is that the strong force QCD, corresponding to the $SU(3)_c$ part of the SM gauge group, grows very strong at energies near $E \sim 1$ GeV and undergoes what is called *confinement*. Under confinement, all $SU(3)_c$ (colour) charged states bind together into colour-neutral objects that are collectively called *hadrons*. The most important of these are the *baryons* and *mesons* that can be matched to the colour-neutral quark operators

$$M \sim \bar{q}^i q'_j \delta^i_j \ , \qquad B \sim q_i q'_j q''_k \epsilon^{ijk} \ , \tag{140}$$

where i and j are colour indices.

The underlying basis of confinement is that the $SU(3)_c$ coupling g_s develops an energy dependence due to quantum (loop) corrections. Writing $g_s(\mu)$ as the effective value at energy $E \sim \mu$, its leading energy dependence is described by the differential equation

$$\frac{dg_s}{dt} = -\frac{b}{(4\pi)^2}\, g_s^3 \ , \tag{141}$$

where $t = \ln(\mu/\mu_0)$ and the constant b depends on the underlying gauge group and the number of quarks charged under it. A nearly unique feature of non-Abelian gauge theories is that this coefficient b can be positive; it is almost always negative for couplings in other quantum field theories in four spacetime dimensions. For the specific case of QCD, b is indeed positive. This implies *asymptotic freedom* in which $g_s(\mu) \to 0$ as $\mu \to \infty$, implying that the theory evoles towards a non-interacting free theory at infinitely high energies. Or put more simply, the strong force actually grows weak as the relevant energy becomes large. In practice, it is found that perturbation theory with quarks and gluons works quite well for computing QCD processes at energies larger than about $E \gtrsim 2$ GeV.

On the flip side, asymptotic freedom also implies that $g_s(\mu)$ grows large as μ becomes small, which occurs around $\mu \sim$ GeV. This suggests a qualitative picture of confinement. Quarks and gluons are weakly-coupled at high energy, but bind very strongly at low energies $E \sim$ GeV as the QCD coupling grows large. Confinement is still not completely understood analytically due to the breakdown of perturbation theory. Instead, our best insight into the mechanics of confinement comes from a combination of computer simulations **??** and detailed experimental measurements [13]. At energies well below $E \sim 1$ GeV, the relevant degrees of freedom are mainly baryons and mesons, and it is possible to formulate perturbative *effective field theories* for them such as *chiral perturbation theory* [31].

Coming back to high energies, where perturbation theory in terms quarks and gluons should work, we are still not able to completely avoid the complicated non-perturbative dynamics of confinement. Even so, it can be shown that asymptotic freedom leads to a *factorization* property in which the perturbative and non-perturbative parts can be split from each other in a (relatively) clean way. For a given high energy process, the perturbative part depends on the specific microscopic process itself while the non-perturbative component is universal and can be obtained from (other) input data.

To illustrate this factorization, consider the process $e^+e^- \to$ hadrons in which the final state includes any combination of mesons and baryons. It is standard practice to plot the ratio

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}\,, \tag{142}$$

as a function of electron-positron CM collision energy $\sqrt{s}$. A collection of data taken for $R(s)$ is shown in Fig. 8. At energies $\sqrt{s} \gtrsim 2$ GeV, it is found that most of the data can be explained well by computing the perturbative processes $e^+e^- \to q\bar{q}$ and summing over all quarks q with mass $m_q < \sqrt{s}/2$. The picture that emerges is that most of the time the underlying collision produces a quark-antiquark pair, with each particle created with much more energy than the scale of confinement. As the quark and antiquark fly off in opposite directions, they each emit soft and collinear QCD radiation that ultimately coalesces into a collimated beam of colour-neutral hadrons called a *jet*. Since these lower-energy *showering* and *hadronization* processes have very little effect on the underlying high-energy *hard* collision, and mainly serve to reprocess the final state each the quark or antiquark, it is a good approximation to compute the total *inclusive* cross section in terms of $q\bar{q}$ production. An important exception to this picture for $R(s)$ occurs near mass thresholds with $\sqrt{s} \sim 2m_q$ where the heavy $q\bar{q}$ pair

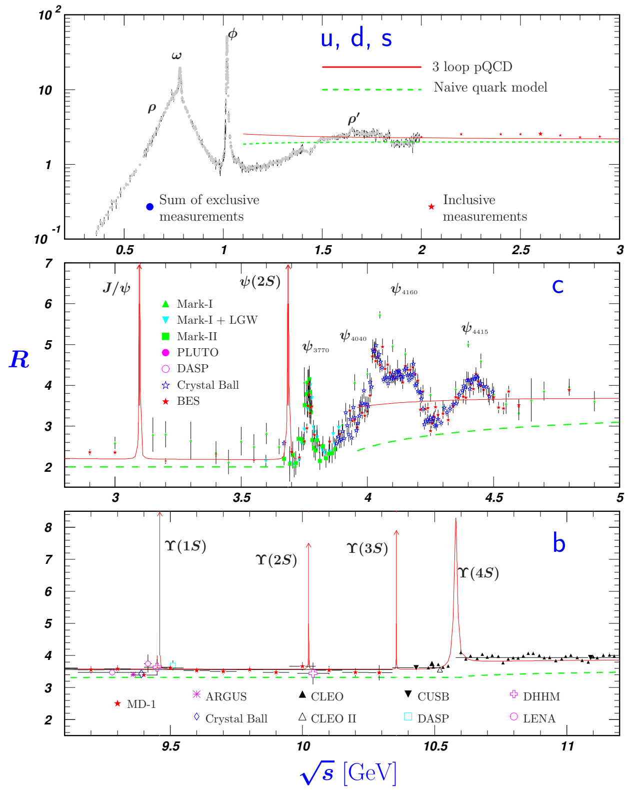

Figure 8: Observed ratios of $R(s) = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ compared with predictions of the quark model, from Ref. [13].

is produced are nearly at rest and can form a bound state. This can be seen in the narrow spikes at certain points in Fig. 8.

Confinement effects can also enter into high-energy processes in the initial state when the objects being collided are hadrons, such as in the proton-proton collisions of the LHC. Here, it is found that highly-energy collisions can be described in terms of collisions between the quarks and gluons within the initial protons. The reason this works is again asymptotic freedom. A proton at the LHC can be thought of as a proton at rest that has been given an enormous boost along the beamline, that we will identify with the z-axis. Treating the proton as a bound state of quarks and gluons, these *parton* constituents will typically have momenta less than $p \sim$ GeV in the proton rest frame. Taking this picture and boosting it along the z-axis by a huge amount, the partons will then have big momenta along the z-axis and much smaller momenta in the directions *transverse* to it. This means that it is a good approximation to keep only the z-component of the initial parton momenta when thinking about collisions. It also motivates a description of the proton constituents in terms of *parton distribution functions* (PDFs). For a given parton type i, $i = g,\, u,\, \bar{u},\, d,\, \bar{d},\, \ldots$, the PDF $f_i^p(x)$ is the (classical) probability that parton-type i will carry a fraction $0 \leq x \leq 1$ of the total proton momentum P. Based on the definition and charge conservation, these must satisfy the sum rules

$$\int_0^1 dx \ \left[f_u^N(x) - f_{\bar{u}}^N(x)\right] = \begin{cases} 2; & N = p \\ 1; & N = n \end{cases} \tag{143}$$

$$f_u^p(x) = f_{\bar{u}}^{\bar{p}}(x) \tag{144}$$

$$1 = \int_0^1 dx \sum_i f_i(x)x \tag{145}$$

The first two results reflect the net quark content of the nucleons (from charge conservation) while the third sum rule corresponds to the partons carrying the momentum of the parent hadron. Not that the PDFs are non-perturbative quantities that must ultimately be determined from data. However, once they are extraacted they can be applied universally to anything else to make predictions.

To illustrate the parton model in action, consider the *Drell-Yan* process $pp \to \ell^+\ell^-$ is (at leading order) illustrated in Fig. 9. The total cross-section in the model is

$$\sigma(pp \to \ell^+\ell^-) = \sum_{ij} \int dx_1 \int dx_2 \ f_i^p(x_1) f_j^p(x_2)\, \hat{\sigma}(q_i(p_1)\bar{q}_i(p_2) \to \ell^-\ell^+) \ , \tag{146}$$

where $p_1 = x_1 P_1$ and $p_2 = x_2 P_2$. In the pp CM frame,

$$p_1 = x_1(E, 0, 0, E) \ , \qquad p_2 = x_2(E, 0, 0, -E) \ , \tag{147}$$

where we have taken the z axis along the direction of the beam. From these expressions we see that the parton-level Mandelstam variable $\hat{s}$ is related to the lab-frame Mandelstam variable s by

$$\hat{s} = (p_1 + p_2)^2 = 2x_1 x_2 \, P_1 \cdot P_2 = x_1 x_2 \, s \ . \tag{148}$$

Note also that even though the collision is taking place in the pp CM frame, this does not coincide in general with the CM frame of the colliding partons. Instead, the parton CM frame has a net boost along the beam direction relative to the lab frame corresponding to the longitudinal momentum $(x_1 - x_2)\sqrt{s}/2$. This makes the kinematics of events at colliders more difficult to reconstruct than if the initial states were fundamental (as opposed to composite) particles. In many cases we focus entirely on the *transverse* momentum $\vec{p}_T$ of the particles that are produced, where $\vec{p}_T$ is the component of a particle's momentum orthogonal (or transverse) to the beam direction.

4 Summary

The SM is great! See Ref. [1] for a more detailed version of this material.

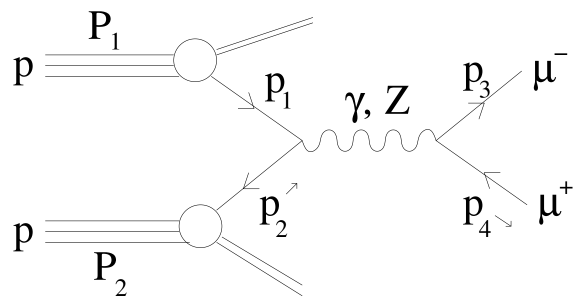

Figure 9: Drell-Yan production of $\mu^+\mu^-$ in a pp collision.

A Appendix: Gamma Matrices

When dealing with Dirac fermions, we often work with 4×4 gamma matrices that satisfy the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \ . \tag{149}$$

These matrices can take different forms corresponding to different choices of basis. The most convenient basis for dealing with the SM is the so-called *chiral basis*. To write the gamma matrices in this basis, it is helpful to generalize to the 2×2 Pauli matrices to

$$\sigma^0 = \mathbb{I}, \quad \sigma^i = \sigma^{1,2,3} \tag{150}$$

with

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{151}$$

Recall that

$$\sigma^i\sigma^j = \delta^{ij}\mathbb{I} + i\epsilon^{ijk}\sigma^k \ . \tag{152}$$

Let us also define

$$\sigma^\mu = (\mathbb{I}, \vec{\sigma}) \ , \qquad \bar{\sigma}^\mu = (\mathbb{I}, -\vec{\sigma}) \ . \tag{153}$$

In terms of these, the 4×4 Dirac matrices in the so-called *chiral basis* are

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} . \tag{154}$$

These satisfy the familiar relation

$$\{\gamma^\mu, \, \gamma^\nu\} = 2\eta^{\mu\nu} \ . \tag{155}$$

It is also standard to define

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\lambda\kappa}\gamma_\mu\gamma_\nu\gamma_\lambda\gamma\kappa \ . \tag{156}$$

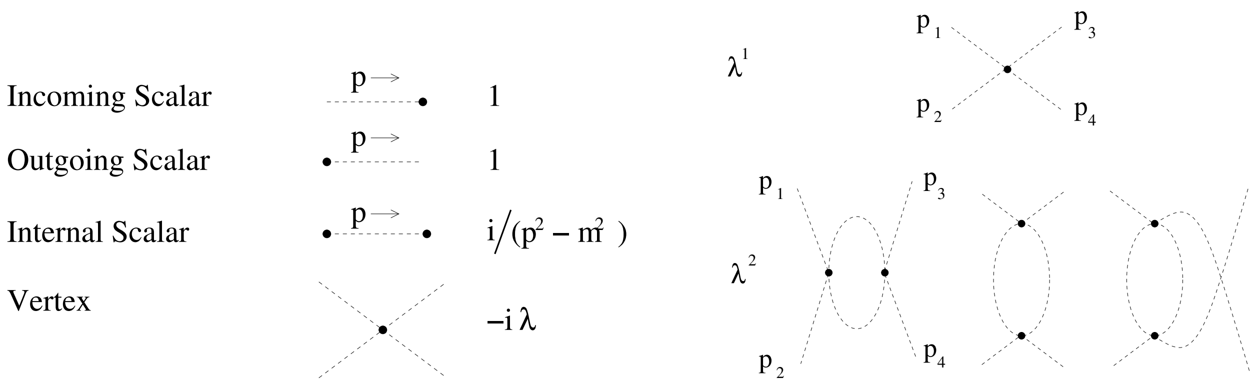

Figure 10: *Left:* Feynman rules for the $\lambda\phi^4$ theory. *Right:* Diagrams for $\phi + \phi \to \phi + \phi$ scattering at linear and quadratic order in powers of λ.

In the chiral representation, one finds

$$\gamma^5 = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} . \tag{157}$$

We will also encounter the chiral projectors

$$P_L = (1-\gamma^5)/2 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix} , \quad P_R = (1+\gamma^5)/2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix} . \tag{158}$$

Note that $P_L^2 = P_L$, $P_R^2 = P_R$, and $P_L P_R = P_R P_L = 0$.

B Appendix: Feynman Rules

I collect here some useful Feynman rules and illustrate them with some simple examples.

B.1 $\lambda\phi^4$ Theory

The Feynman rules for this theory are given in the left panel of Fig. 10. For a given process, you should write down all possible diagrams for it (to the order in λ you want) following the steps outlined in Sec. 1.1.5. External legs get a factor of unity, internal scalars get a factor of $i/(p^2 - m^2)$ which is sometimes called the *propagator*, and the vertex is $-i\lambda$.

The simplest scattering process in this theory is $\phi(p_1) + \phi(p_2) \to \phi(p_3) + \phi(p_4)$. The Feynman diagrams for the contributions to the scattering amplitude to quadratic order in powers of λ are shown in the right panel of Fig. 10. Using the Feynman rules, the scattering amplitude is

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \dots \tag{159}$$

To leading non-trivial order in powers of λ, we have simply

$$-i\mathcal{M} \simeq -i\mathcal{M}_1 = -i\lambda . \tag{160}$$

This diagram just has the vertex and external legs, but since the external legs all give factors of unity, the amplitude is equal to the vertex value. Easy! But things get more complicated at higher orders.

B.2 QED

The Feynman rules for quantum electrodynamics (QED) with a fermion of mass m and charge Q are given in Fig. 11. In contrast to the scalar theory, external legs for fermions or vectors now get a factor of the outgoing spin polarization state. Fermion legs also come with arrows that denote the direction of charge flow. For external legs, an arrow pointing in on an initial state denotes an incoming fermion (and factor of u), an arrow pointing out on an initial state denotes an incoming antifermion (and factor $\bar{v}$), an arrow pointing out on an outgoing state denotes and outgoing fermion (and factor of $\bar{u}$), and an arrow point in on outgoing state denotes an outgoing anti-fermion (and factor of v). In practice, to evaluate a diagram choose a fermion line in it, work backwards along the fermion arrows and collect the relevant factors as you go along. Do this for all fermion lines, and then add the factors for photon lines.

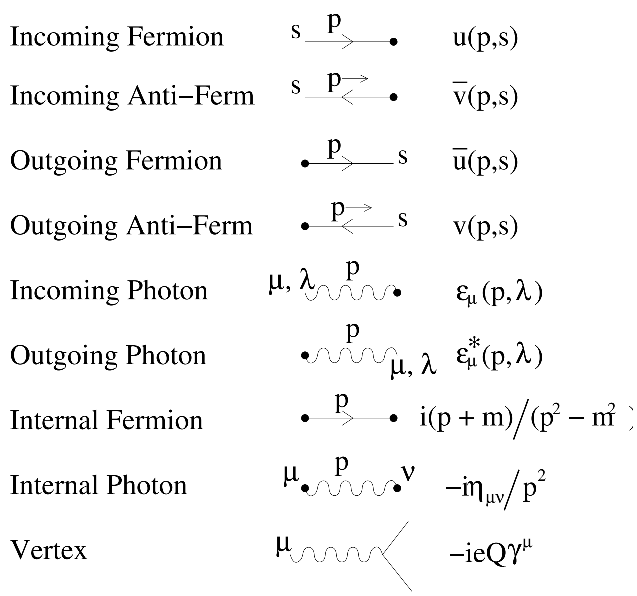

Figure 11: Feynman rules for QED.

B.3 Non-Abelian Gauge Theory

Feynman rules for non-Abelian gauge theories are given in Fig. 12 for the case of a fermion transforming in a rep of the gauge group with generator t^a. These rules are very similar to QED but with a few additional factor. The main changes related to the non-Abelian nature of the theory are the new vector boson self-interaction vertices and the generator factors of

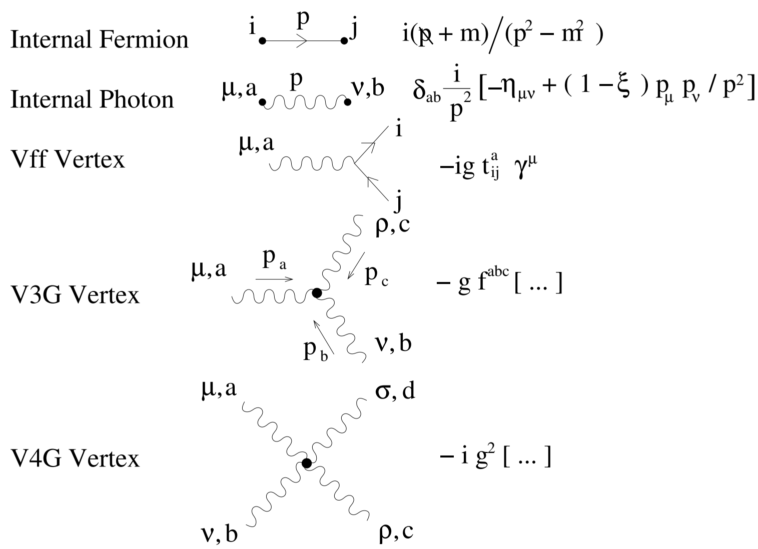

Figure 12: Feynman rules for a non-Abelian gauge theory.

t^a in vertices with matter fields. Some additional details on how to use these Feynman rules are given in the SM discussion below.

B.4 The Standard Model

To compute within the SM, it is standard to work in the mass eigenstate basis. To simplify the notation, we will drop the primes on these states that we had been using to distinguish them from the gauge eigenstates. It is also customary to assemble the 2-component SM fermions into 4-component objects so that we can use our tricks for γ^μ matrices. For this, we write

$$u = \begin{pmatrix} u_L \\ u_R \end{pmatrix}, \quad d = \begin{pmatrix} d_L \\ d_R \end{pmatrix}, \quad e = \begin{pmatrix} e_L \\ e_R \end{pmatrix}, \quad \nu_L = \begin{pmatrix} \nu_L \\ 0 \end{pmatrix}. \tag{161}$$

To isolate the L or R components, we apply the chiral projectors P_L and P_R to these 4-component fermions in the Feynman rules.

The propagators for the SM fermions and the Higgs boson are the same as we had before, and are shown in Fig. 13. For vectors, the propagators are (for momentum p)

$$A_\mu \to A_\nu : \qquad \frac{i}{p^2}(-\eta_{\mu\nu}) \tag{162}$$

$$G^a_\mu \to G^b_\nu : \qquad \frac{i}{p^2}\left[-\eta_{\mu\nu} + (1-\xi)p_\mu p_\nu/p^2\right]\delta^{ab} \tag{163}$$

$$Z_\mu \to Z_\nu : \qquad \frac{i}{p^2 - m_Z^2}\left(-\eta_{\mu\nu} + p_\mu p_\nu/m_Z^2\right) \tag{164}$$

$$W^\pm_\mu \to W^\pm_\nu : \qquad \frac{i}{p^2 - m_W^2}\left(-\eta_{\mu\nu} + p_\mu p_\nu/m_W^2\right) \tag{165}$$

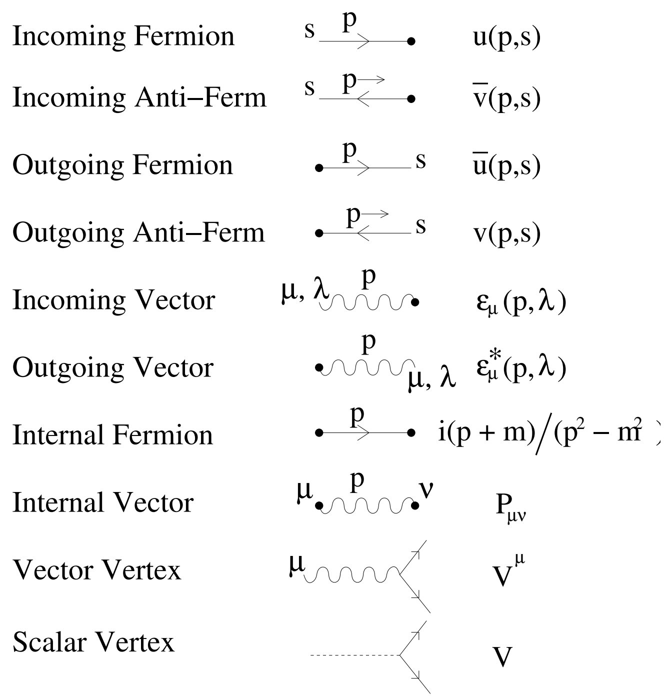

Figure 13: Feynman rules for the Standard Model.

The factor ξ in the gluon propagator depends on the choice of gauge and should cancel out in any physically observable quantity. The $W^{\pm}$ and Z^0 propagators correspond specifically to our choice of *unitary gauge*, and they describe the propagation of a massive vector.[13]

Spin polarization factors for external fermion lines in a Feynman diagram are identical to those we discussed for QED and general non-Abelian gauge theories. External vector lines pick up a polarization 4-vector $\epsilon^{\mu}(p,\lambda)$, as shown in Fig. 13, where p is the momentum of the vector and λ labels the polarization state. Massive and massless vectors have different numbers of polarization states. The massless photon and gluons have *two* physical transverse polarizations, while the massive $W^{\pm}$ and Z^0 vectors have three. Independent of whether a vector is massive or massless, we always have

$$\epsilon_{\mu}(p,\lambda)\,p^{\mu} = 0\ . \tag{166}$$

These properties have important consequences for evaluating Feynman diagrams. In many cases we only care about the unpolarized cross-section, where the final polarizations are summed over and the initial polarizations are averaged. These spin sums can often be

[13] They take different forms in other gauges.

simplified using (partial) completeness relations. For the SM vectors, we have

$$W_\mu,\ Z_\mu: \qquad \sum_{\lambda=1}^{3}\epsilon_\mu(p,\lambda)\epsilon_\nu^*(p,\lambda) = -\eta_{\mu\nu} + p_\mu p_\nu/m^2 \tag{167}$$

$$A_\mu: \qquad \sum_{\lambda=1}^{2}\epsilon_\mu(p,\lambda)\epsilon_\nu^*(p,\lambda) = -\eta_{\mu\nu} + (\text{stuff you can ignore}) \tag{168}$$

$$G_\mu: \qquad \sum_{\lambda=1}^{2}\epsilon_\mu(p,\lambda)\epsilon_\nu^*(p,\lambda) = -\eta_{\mu\nu} + (\text{stuff you can't ignore}) \tag{169}$$

The non-ignorable stuff for the gluon polarization sum is related to the presence of non-decoupling ghost fields in the theory. In this case, it usually easiest to choose an explicit set of transverse polarization vectors satisfying

$$\epsilon(p,\lambda)\cdot\epsilon^*(p,\lambda') = \delta_{\lambda,\lambda'}, \qquad p\cdot\epsilon(p,\lambda) = 0, \qquad (1,0,0,0)\cdot\epsilon(p,\lambda) = 0\ . \tag{170}$$

Of course, explicit polarizations can also be used for the other vectors.

There are lots of interaction vertices in the SM, and they are straightforward to work out from the Lagrangian. We will collect only the fermion-vector couplings here. Comparing to the general notation in Fig. 13, the fermion-photon vertex for $\psi_j \to A_\mu\psi_i$ is

$$V^\mu = -ieQ\gamma^\mu\,\delta_{ij}\ , \tag{171}$$

where i, j label the colour of the fermion ψ (with $i = j = 1$ if the fermion is uncoloured). For the gluon, the basic vertex for $\psi_j \to G^a_\mu\psi_i$ is

$$V^\mu = -ig_s\gamma^\mu t^a_{ij}\ , \tag{172}$$

where t^a_{ij} is the representation matrix corresponding to the $SU(3)_c$ rep of the fermion ψ_i. Note that for a trivial representation, $t^a_{ij} = 0$ and the vertex vanishes. The Z^0 vertex for $\psi_j \to Z_\mu\psi_i$ is

$$V^\mu = -i\bar{g}\gamma^\mu\left[(t^3 - Qs_W^2)P_L + (0 - Qs_W^2)P_R\right]\,\delta_{ij}\ . \tag{173}$$

Fermion projectors have been used here to isolate the different couplings of the Z^0 to the left- and right-handed components of the 4-component fermions we are working with. For the $W^\pm$, we have for $\psi_{Bj} \to W^-_\mu\psi'_{Ai}$ (where A, B are the flavour indices of the fermion)

$$V^\mu = -i\frac{g}{\sqrt{2}}\gamma^\mu P_L V^{(CKM)}_{AB}\,\delta_{ij}\ . \tag{174}$$

Note that here ψ is the lower component of an $SU(2)_L$ doublet while ψ' is an upper component. For $\psi'_A \to W^+_\mu\psi_B$ one gets the same vertex but with $V^{(CKM)\dagger}_{AB}$. Note also that except for the gluon coupling, the incoming and outgoing colour states at a vector vertex are the same, corresponding to the δ_{ij} factors. Similarly, the flavours of the incoming and outgoing fermions are identical except for the $W^\pm$ couplings and so we have not included flavour indices in the other vertices. The vertex for a fermion ψ coupling to the Higgs boson is

$$V = -i\frac{m_\psi}{\sqrt{2}v}\,\delta_{ij}\ , \tag{175}$$

where m_ψ is the fermion mass. This coupling is also diagonal in flavour (and colour) space.

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